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The Schrödinger equation written in the second quantization formalism: derivation from first principles

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Abstract

Courses about Quantum Mechanics are generally developed at the level of the mathematical formalism, its syntactic level, seldom treating the interpretation of the theory, its semantic level. In the majority of cases, this first formal approach is not followed by any other course addressing the numerous interpretations of the theory. It has become apparent that this strategy leads to a deficit in the professional formation of the students, exceedingly driving them to technical areas of application of Quantum Mechanics. However, applications such as those on quantum computing and entanglement, as they become more and more encrusted in the deepest foundations of the theory, are asking more and more for the development of this kind of ability. In this paper we develop an approach to deal with second quantization in the realm of Quantum Mechanics. We show that the second quantization Schrödinger equation can be mathematically derived from a classical Hamiltonian written in the phase space (Q, P)*, obtained by a canonical transformation upon the original phase space* (q, p)*, using an axiomatic quantization procedure developed elsewhere. We apply this quantization process to a bosonic system, modelled by the harmonic oscillator problem. We then make reverse engineering to show that Schwinger's second quantization approach to fermionic systems furnishes the path to derive a Schrödinger equation explicitly written in terms of the usual momenta and positions operators* $(\hat{q}, \hat{p})$ *for such systems. Finally, we use these mathematical developments to address the commonly accepted statement that 'the spin has no classical analog'.*

Keywords: *Second Quantization. Canonical Transformations. Quantum Mechanics.*

I. Introduction

In his notorious work, Thomas Kuhn introduced the technical concept of *manual* as meaning a text devoted to develop the technical aspects of a theory (generally those related to its mathematical structure). In a sense, manuals are important to give the students proficiency in the manipulative aspects of the formal structure of the theory. However, manuals, important as they really are, cannot be considered an exhausting assessment of the theory, in what it respects to its teaching, because manuals are written in such a way to precisely hide the theory's eventual problems or points of disagreement.

It is assumed that the students should first understand the syntactic apparatus of the theory, become proficient in its use, to finally become in a position to understand its most intricate interpretation problems. This, of course, can be questioned on the grounds that even manuals present *some* interpretation of Quantum Mechanics, even if they do that in a minimalist perspective. Be it as it may, the fact is that after this first encounter with the theory, when the student is finally in position to get into contact with the semantic apparatus of the theory, there are quite a few courses devoted to that, and they are rarely considered as an integral part of the formation of the student (courses on the interpretation of Quantum Mechanics are ubiquitously of the elective type, not the obligatory one).

On the other hand, many new applied fields of Quantum Mechanics are becoming deeply grounded in its interpretation structure. Elements of entanglement and quantum computation, to cite a few, are receiving much attention presently and they are calling for a different formation of our students. Nowadays Quantum Mechanics is showing that the mere formal assessment, with some minimal interpretation, are quickly finding its limits. It would then be important to take a different path in the teaching of this theory.

Most teachers, formed within the rather technical perspective previously mentioned don't feel themselves in condition to make a more philosophical approach of some aspects of the interpretation of Quantum Mechanics. However, they may feel comfortable with an approach that pinpoints aspects of the interpretation of Quantum Mechanics precisely by following closely its formal structure.

The objective of this paper is to discuss the common sense statement that "the quantum mechanical half integral spin has no classical analog" and its implications. To do that, we will address another important (and rather obscure) notion of the quantum mechanical *formalism*, regarding the role that canonical transformation play within it. Our discussions will be based on some aspects of the second quantization formalism.

II. Second Quantization

The usual Schrödinger equation (hereafter USE) is generally written in terms of a Hamiltonian operator $\hat{H}$ written in terms of position and momentum operators $(\hat{q}, \hat{p})$. This Hamiltonian operator is obtained from the classical Hamiltonian function $H(q, p)$ defined on the phase-space (q, p) by the substitution $\hat{q} = q$ and $\hat{p} = -i\hbar\partial/\partial q$.

In the literature it is usual to find, after the presentation of USE, the derivation of a second quantization Schrödinger equation (hereafter SQSE) in terms of creation and annihilation operators $(a^{\dagger}, a)$. These operators are introduced as a linear transformation of the original

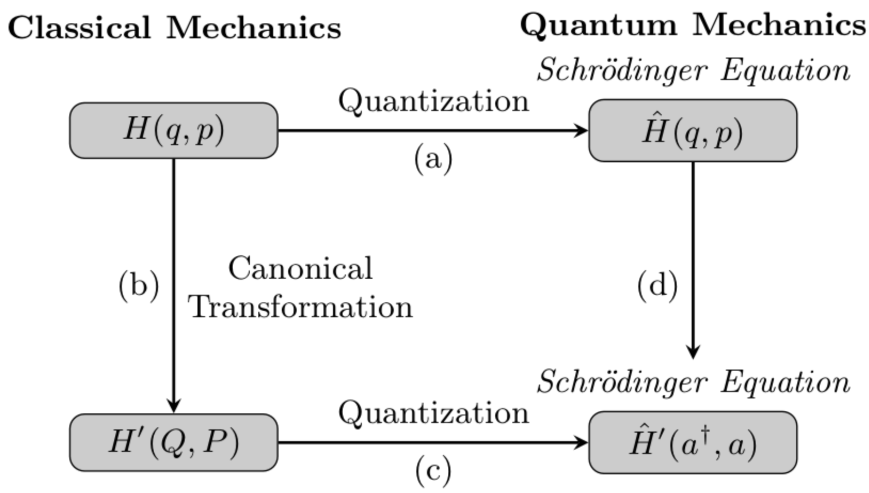

Figura 1: *Quantization processes applied after a canonical transformation to reveal the equivalence between the Schrödinger equation, written in terms of the usual position and momentum variable* (q,p)*, and the creation and annihilation operators* $(a^\dagger,a)$*.*

$(\hat{q},\hat{p})$ operators. This transformation, however, is never the object of criticism, despite the fact that it involves a transformation on the position and the momentum, which would characterize a canonical transformation in the realm of Classical Mechanics.

Interesting enough, the transformation normally used in textbooks (GASIOROVICZ, 2003) to pass from USE to SQSE *does not* correspond to a canonical transformation, when considered from the point of view of Classical Mechanics. This puts an interrogation mark on the possible physical equivalence between the original and derived systems described by such equations. Indeed, from the point of view of Classical Mechanics, canonical transformations are the only way to guarantee that we keep ourselves within the boundaries of the same physical system, whenever the original (p,q) variables are taken into new (P,Q) variables.

Even if we are in the realm of Quantum Mechanics, within which general canonical transformations are disputable, we are surely not free to make transformations out of our own device on the variables (q,p) – in the Classical Hamiltonian to proceed with quantization – nor on the corresponding operators $(\hat{q},\hat{p})$. This is most certainly true if we keep ourselves within the scope of *linear* canonical transformations (KIM & NOZ, 1991), which is the type of transformation that takes USE into SQSE.

One of the objectives of this paper is to show how to put the usual second quantization formalism of a bosonic system on the sound grounds of canonical transformations. To accomplish that, we first show how to *mathematically derive* USE from a classical Hamiltonian and from first principles, based on a very simple axiomatic approach – the quantization process (process (a) in Figure 1). We then prove that we can begin with a usual harmonic oscillator system (a paradigm for bosonic systems) written within Classical Mechanics, make a canonical transformation in the Classical "side" (transformation (b) in Figure 1), and then use the same quantization process, rephrased in the new variables (quantization process (c) in Figure 1) to find the correct SQSE in the Quantum "side" (see Figure 1). The transformation (d) in Figure 1 is the usual way by which one gets SQSE from USE

(GASIOROWICZ, 2003).

In the context of the second quantization formalism, we then apply the same method to Schwinger's representation of fermionic systems in terms of bosonic creation and annihilation operators to find, by reverse engineering, the Schrödinger equation in terms of the variables (q, p) for a half-integral spin particle (the reverse path: (c) $\rightarrow$ (b) $\rightarrow$ (a) in Figure 1). If such an equation is possible, then the usual assumption that "there is no classical analog for the spin" cannot be sustained. As already argued in the previous section, this is the other and main objective of this paper. We follow a line of reasoning that would give traditional teachers all the means to address this semantic aspect of the theory departing from, and closely linked with, its syntactic apparatus.

This work is organized as follows. In Sec. 2 we review the basic formalism of the axiomatic derivation to show that we can mathematically derive the usual Schrödinger equation for any physical system from the associated classical Hamiltonian $H(q, p)$ (process (a) in Figure 1). This is done to show the soundness of the quantization procedure. This quantization process will not be of our inquiry at this point, but will be addressed in all its depth in a future work.

In Sec. 3 we define the canonical transformation $(Q, P) = F(q, p)$ to rewrite the *classical* Hamiltonian in the new variables (Q, P) (process (b) in Figure 1). We then apply the same quantization method to the resulting Hamiltonian $H'(Q, P)$ to rewrite the original Schrödinger equation in the second quantization formalism (process (c) in Figure 1). We then show that this process gives the usual method of second quantization (process (d) in Figure 1) soundness.

In Sec. 4 we use the same quantization process for fermionic systems. These systems can be written in terms of bosonic creation and annihilation operators using the so called Schwinger's representation (SCHWINGER, 1952). We thus find a Schrödinger equation in terms of momentum and position operators for fermionic systems. We then stress the connection between these results and those obtained in previous works (OLAVO & FIGUEIREDO, 1999) about the representation of the half-integral spin in the classical phase space and discuss the issue of a classical counterpart for the quantum mechanical half-integral spin.

Section 5 is devoted to our conclusion.

III. The quantization method

The following axioms allow us to mathematically derive the Schrödinger Equation for any quantum mechanical physical system to which one can furnish a Hamiltonian function (OLAVO, 2016).

Axiom 1: the joint phase-space probability density function related to any isolated Quantum Mechanical system is such that

$$\frac{dF(q, p; t)}{dt} = 0. \quad (1)$$

Axiom 2: the characteristic function of $F(q,p;t)$, written as the Fourier transform

$$Z(q,\delta q;t) = \int_{-\infty}^{+\infty} F(q,p;t) e^{ip\delta q/\hbar} dp\,, \tag{2}$$

can be decomposed as

$$Z(q,\delta q;t) = \psi^* \left(q - \frac{\delta q}{2};t \right) \psi \left(q + \frac{\delta q}{2};t \right). \tag{3}$$

Note that Z is the usual characteristic function as defined in the particular context of statistical physics, and in the context of statistics, more generally.

We now assume a Hamiltonian function

$$H(q,p) = \frac{p^2}{2m} + V(q)\,, \tag{4}$$

for a single particle with mass m, kinetic energy $p^2/2m$, moving under a one dimensional potential $V(q)$.

Using axiom 1, we can write

$$\frac{\partial F}{\partial t} + \dot{q}\frac{\partial F}{\partial q} + \dot{p}\frac{\partial F}{\partial p} = \frac{\partial F}{\partial t} + \frac{p}{m}\frac{\partial F}{\partial q} - \frac{\partial V}{\partial q}\frac{\partial F}{\partial p} = 0\,, \tag{5}$$

where the second equality follows from Hamilton's canonical equations of motion for $H(q,p)$ (GOLDSTEIN, 2011).

By axiom 2, the characteristic function of this system can be written through the Fourier transform (2), which can be used to take (5) into

$$-\frac{\hbar^2}{m}\frac{\partial^2 Z}{\partial q \partial(\delta q)} + \delta q \frac{\partial V}{\partial q} Z = i\hbar \frac{\partial Z}{\partial t}\,. \tag{6}$$

We now use the fact that the $\psi(q;t)$, whose product gives $Z(q,\delta q;t)$, are complex functions. Thus, they can be written in polar form as

$$\psi(q;t) = R(q;t) e^{\frac{i}{\hbar}S(q;t)}\,, \tag{7}$$

where R and S are real functions of q and t. Thus,

$$\psi \left(q + \frac{\delta q}{2};t \right) = R \left(q + \frac{\delta q}{2};t \right) e^{\frac{i}{\hbar}S\left(q+\frac{\delta q}{2};t\right)}\,. \tag{8}$$

Using a Taylor expansion up to second order in the parameter δq for R and S we find

$$\psi\left(q+\frac{\delta q}{2};t\right)\approx\left[R(q;t)+\frac{\delta q}{2}\frac{\partial R(q;t)}{\partial q}+\frac{(\delta q)^2}{8}\frac{\partial^2 R(q;t)}{\partial q^2}\right] \times\exp\left\{\frac{i}{\hbar}\left[S(q;t)+\frac{\delta q}{2}\frac{\partial S(q;t)}{\partial q}+\frac{(\delta q)^2}{8}\frac{\partial^2 S(q;t)}{\partial q^2}\right]\right\}. \tag{9}$$

From axiom 2 we write the characteristic function as

$$Z(q,\delta q;t)=\psi^*\left(q-\frac{\delta q}{2};t\right)\psi\left(q+\frac{\delta q}{2};t\right)= \left\{R^2(q;t)+\left(\frac{\delta q}{2}\right)^2\left[R(q;t)\frac{\partial^2 R(q;t)}{\partial q^2}-\left(\frac{\partial R(q;t)}{\partial q}\right)^2\right]\right\} \times\exp\left\{\frac{i\delta q}{\hbar}\frac{\partial S(q;t)}{\partial q}\right\}, \tag{10}$$

where we have assumed that the expansion should be done only up to second order – the deep justification for this assumption is connected to the Central Limit Theorem. This was shown elsewhere (OLAVO, 2016) and will not be of our concern in this paper.

Finally, with Z already expressed in terms of its decomposition, we put it back into equation (6) and split the result into real and imaginary parts to get

$$\frac{\partial R^2(q;t)}{\partial t}+\frac{\partial}{\partial q}\left[\frac{R^2(q;t)}{m}\frac{\partial S(q;t)}{\partial q}\right]=0 \tag{11}$$

and

$$\frac{\partial S(q;t)}{\partial t}+V(q)+\frac{1}{2m}\left(\frac{\partial S(q;t)}{\partial q}\right)^2-\frac{\hbar^2}{2mR(q;t)}\frac{\partial^2 R(q;t)}{\partial q^2}=0. \tag{12}$$

These last two equations are exactly the ones we get if we put $\psi(q;t)$, written as in (7), into the usual Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(q;t)}{\partial q^2}+V(q)\psi(q;t)=i\hbar\frac{\partial\psi(q;t)}{\partial t}. \tag{13}$$

and separate the result into real and imaginary parts. This result shows that equations (12) and (11) are mathematically equivalent to (13). We thus have a quite general method to mathematically derive the Schrödinger equation.

IV. Quantization of a Bosonic System

The most elementary bosonic Hamiltonian is the one related to the harmonic oscillator system. Since this system is seminal to treat bosonic many-body systems, we focus here on its quantization process. We already know the result of applying axioms 1 and 2 to the usual harmonic oscillator system (quantization process (a) in Figure 1). We may now try to find the result of the same quantization process applied to a canonically transformed Hamiltonian.

IV.1. Canonical Transformation

Since Hamilton's equations are invariant under canonical transformations, our interest resides on knowing the effect of the transformation $(q,p) \to (Q,P)$ on the harmonic oscillator system for transformations of the type

$$Q = \mu\,(aq + ibp) \quad ; \quad P = \nu\,(aq - ibp)\,, \tag{14}$$

and their respective inverse transformations

$$q = \frac{1}{2a}\left(\frac{Q}{\mu} + \frac{P}{\nu}\right) \quad ; \quad p = \frac{1}{2ib}\left(\frac{Q}{\mu} - \frac{P}{\nu}\right), \tag{15}$$

with μ, ν, a and b all known (generally complex) constants yet to be determined. We remind that a sufficient condition for the transformation (14) to be canonical is given by the relations (GOLDSTEIN, 2011)

$$\begin{gathered}\frac{\partial Q}{\partial q} = \frac{\partial p}{\partial P}, \qquad \frac{\partial Q}{\partial p} = -\frac{\partial q}{\partial P}, \\ \frac{\partial P}{\partial q} = -\frac{\partial p}{\partial Q} \quad \text{and} \quad \frac{\partial P}{\partial p} = \frac{\partial q}{\partial Q}.\end{gathered} \tag{16}$$

The classical Hamiltonian of the harmonic oscillator can be written as

$$H_C = \omega\left(\frac{p^2}{2m\omega} + \frac{m\omega}{2}q^2\right), \tag{17}$$

where m and ω are the mass and frequency of the system, as usual. Relations (16) and (14) imply that

$$\mu\nu = -\frac{1}{2abi}. \tag{18}$$

Setting a and b as

$$a = \sqrt{\frac{m\omega}{2}} \quad \text{and} \quad b = \sqrt{\frac{1}{2m\omega}}, \tag{19}$$

we have from (18) the conditions that assure the canonical character of our transformations (14) as

$$ab = \frac{1}{2} \quad \text{and} \quad \mu\nu = i\,. \tag{20}$$

Then, transformation (14) can be rewritten as

$$\begin{aligned} Q &= \mu\left(\sqrt{\frac{m\omega}{2}}q + i\sqrt{\frac{1}{2m\omega}}p\right) \\ P &= \nu\left(\sqrt{\frac{m\omega}{2}}q - i\sqrt{\frac{1}{2m\omega}}p\right). \end{aligned} \tag{21}$$

With this canonical transformation our classical Hamiltonian becomes

$$H' = -i\omega QP, \tag{22}$$

and the Poisson's brackets for our new canonical coordinates are

$$\{Q, H'\} = -i\omega\{Q, QP\} = -i\omega Q = \frac{dQ}{dt}, \tag{23}$$

and

$$\{P, H'\} = -i\omega\{P, QP\} = i\omega P = \frac{dP}{dt}. \tag{24}$$

These last two equations could be derived directly from Hamilton's equations

$$\frac{\partial H'}{\partial P} = \dot{Q} = -i\omega Q \quad \text{and} \quad \frac{\partial H'}{\partial Q} = -\dot{P} = -i\omega P. \tag{25}$$

In terms of a new variable

$$\tau = it, \tag{26}$$

they generate the differential equations

$$\frac{dQ}{d\tau} = -\omega Q \quad \text{and} \quad \frac{dP}{d\tau} = \omega P, \tag{27}$$

and the respective solutions

$$Q = Q_0 e^{-i\omega t} = Q_0 e^{-\omega\tau} \quad \text{and} \quad P = P_0 e^{i\omega t} = P_0 e^{\omega\tau}, \tag{28}$$

with Q_0 and P_0 constants determined by the initial conditions of our system.

The imaginary time τ in (26) appears quite naturally since the transformed Hamiltonian in (22) is also an imaginary quantity, and the Hamiltonian is the generator of time translations (GOLDSTEIN, 2011).

IV.2. Quantization Process

Axiom 1 for the transformed Hamiltonian is given by

$$\frac{\partial F}{\partial t} + \dot{Q}\frac{\partial F}{\partial Q} + \dot{P}\frac{\partial F}{\partial P} = 0 \tag{29}$$

and from our equations of motion (23) and (24) we get

$$\frac{\partial F}{\partial t} - i\omega\left(Q\frac{\partial F}{\partial Q} - P\frac{\partial F}{\partial P}\right) = 0, \tag{30}$$

so that, using our new variable $\tau = it$, we can rewrite the previous equation as

$$\frac{\partial F}{\partial \tau} - \omega\left(Q\frac{\partial F}{\partial Q} - P\frac{\partial F}{\partial P}\right) = 0. \tag{31}$$

In order to find an equation for $Z(Q,\delta Q;\tau)$, we first write the characteristic function in axiom 2 in terms of the new variables as

$$Z(Q,\delta Q;\tau) = \int_{-\infty}^{\infty} F(Q,P;\tau)e^{iP\delta Q/\hbar}dP. \tag{32}$$

Multiplying expression (31) by $\exp\{iP\delta Q/\hbar\}$ and integrating in dP we get, after some straightforward calculations,

$$\frac{\partial Z}{\partial \tau} - \omega\left[Q\frac{\partial Z}{\partial Q} + Z + \delta Q\frac{\partial Z}{\partial(\delta Q)}\right] = 0. \tag{33}$$

In terms of the new variables, axiom 2 says that the characteristic function must be written as the product

$$Z(Q,\delta Q;\tau) = \psi^*\left(Q - \frac{\delta Q}{2};\tau\right)\psi\left(Q + \frac{\delta Q}{2};\tau\right). \tag{34}$$

Repeating the same steps presented in section 2 to find USE we get the two equations

$$\frac{\partial R^2}{\partial \tau} - \frac{\partial\left(\omega Q R^2\right)}{\partial Q} = 0 \tag{35}$$

and

$$\frac{\partial S}{\partial \tau} - \omega Q\frac{\partial S}{\partial Q} = 0. \tag{36}$$

It is easy to show that these last two equations can be derived from the Schrödinger equation (we get back to the real time t)

$$-\hbar\omega\left(Q\frac{\partial\psi}{\partial Q} + \frac{1}{2}\psi\right) = i\hbar\frac{\partial\psi}{\partial t}, \tag{37}$$

or

$$\omega\left[Q\frac{\partial\psi}{\partial Q} + \frac{1}{2}\psi\right] = -i\frac{\partial\psi}{\partial t}, \tag{38}$$

written in terms of the operators $\hat{Q} = Q$ and $\hat{P} = -i\hbar\partial/\partial Q$.

Indeed, if we represent $\psi(Q,t)$ in its polar representation

$$\psi(Q,t) = R(Q;t)\exp\left[iS(Q;t)/\hbar\right], \tag{39}$$

equation (38) becomes

$$\begin{aligned} \omega Q\frac{\partial R}{\partial Q} + \frac{i\omega}{\hbar}QR\frac{\partial S}{\partial Q} + \frac{\omega}{2}R = -i\frac{\partial R}{\partial t} + \frac{1}{\hbar}R\frac{\partial S}{\partial t} = \\ = \frac{\partial R}{\partial \tau} + \frac{i}{\hbar}R\frac{\partial S}{\partial \tau}. \end{aligned} \tag{40}$$

Splitting this last equation in both real and imaginary parts we find the two equations (36) and (35), showing that (38) is, indeed, the correct Schrödinger equation.

If we want to pass to the creation and annihilation operators formalism, we may simply write $a^\dagger = e^{i\omega t}Q$ and $a = e^{-i\omega t}\partial/\partial Q$ to rewrite (38) as

$$\hbar\omega\left(a^\dagger a + \frac{1}{2}\right)\psi = -i\hbar\frac{\partial\psi}{\partial t}, \tag{41}$$

or, noting that $\psi(Q,t) = R(Q;t)\exp(iEt/\hbar)$ [the absence of the minus sign comes from the minus sign in (22)], as

$$\hbar\omega\left(a^\dagger a + \frac{1}{2}\right)\psi = E\psi. \tag{42}$$

With these operators we obtain the commutator

$$\left[a,a^\dagger\right]\psi = \frac{\partial(Q\psi)}{\partial Q} - Q\frac{\partial\psi}{\partial Q} = \psi \Rightarrow \left[a,a^\dagger\right] = 1. \tag{43}$$

Furthermore, $\psi(Q;t) = Q^n\exp(iEt/\hbar)$ is a solution for (42) if $E = E_n = \left(n+\frac{1}{2}\right)\hbar\omega$, so that the complete solution can be written as

$$\psi_n(Q,t) = Q^n e^{i\left(n+\frac{1}{2}\right)\omega t}\phi_0 = \left(a^\dagger\right)^n\phi_0 e^{i\omega t/2}, \tag{44}$$

where ϕ_0, in this representation, is a constant such that $a\phi_0 = \partial\phi_0/\partial Q = 0$.

In terms of the operators a and $a^\dagger$, the transformed quantum mechanical Hamiltonian operator becomes

$$\hat{H}' = \left(Q\frac{\partial}{\partial Q} + \frac{1}{2}\right)\hbar\omega = \left(a^\dagger a + \frac{1}{2}\right)\hbar\omega, \tag{45}$$

and the commutators of those new operators with $\hat{H}'$ are given by

$$\begin{aligned} [a,\hat{H}']\psi_n = \left[\frac{\partial}{\partial Q},\hat{H}'\right]\psi_n = \\ = \left(\frac{\partial}{\partial Q}Q\frac{\partial\psi_n}{\partial Q} - Q\frac{\partial}{\partial Q}\frac{\partial\psi_n}{\partial Q}\right) = \frac{\partial\psi_n}{\partial Q} = a\psi_n, \end{aligned} \tag{46}$$

and

$$\left[a^{\dagger},\hat{H}'\right] = [Q,\hat{H}']\,\psi_n = \\ = \left(QQ\frac{\partial\psi_n}{\partial Q} - Q\frac{\partial}{\partial Q}Q\psi_n\right) = -Q\psi_n = -a^{\dagger}\psi_n\,, \tag{47}$$

This means that

$$a^{\dagger}\hat{H}' - \hat{H}'a^{\dagger} = -a^{\dagger} \Rightarrow \hat{H}'a^{\dagger} = a^{\dagger}\left(\hat{H}' + 1\right), \tag{48}$$

and

$$\hat{H}'a - a\hat{H}' = a \Rightarrow \hat{H}'a = a(\hat{H}' - 1)\,, \tag{49}$$

showing that the operators $a^{\dagger}$ and a really behave as, respectively, the creation and annihilation operators in the usual formalism of quantum mechanics (BAYM, 1969). This result could also be obtained by simply applying directly the operators a and $a^{\dagger}$ to a solution of the type (44).

IV.3. Representations

We may now take a look at the two representations resulting from the quantization of two classical Hamiltonian functions connected by a canonical transformation.

We know that the solution of the problem related to the Hamiltonian (17) is simply the unnormalized function

$$\psi_n(q;t) = H_n(q)\exp\left(-\frac{m\omega}{2\hbar}q^2\right)\exp\left[-i\left(n+\frac{1}{2}\right)\omega t\right], \tag{50}$$

where $H_n(q)$ are the Hermite polynomials of order n. The ground state solution is obtained making $n = 0$ such that

$$\psi_0(q;t) = \exp\left(-\frac{m\omega}{2\hbar}q^2\right)\exp\left(-\frac{i\omega t}{2}\right). \tag{51}$$

On the other hand, the unnormalized solution of the same problem, related to the canonically transformed Hamiltonian (22) is given by

$$\psi_n(Q,t) = Q^n e^{i\left(n+\frac{1}{2}\right)\omega t}\phi_0 = \left(a^{\dagger}\right)^n \phi_0 e^{i\omega t/2}. \tag{52}$$

The connection between the two representations is given by writing ϕ_0, which is a constant in the (Q,P) representation in the previous equation, as

$$\phi_0 = \exp\left(-\frac{m\omega}{2\hbar}q^2\right), \tag{53}$$

such that, using (21), one gets

$$\psi_n(q;t) = \left(\sqrt{\frac{m\omega}{2}}q + \sqrt{\frac{1}{2m\omega}}\hbar\frac{d}{dx}\right)^n \exp\left(-\frac{m\omega}{2\hbar}q^2\right)\exp\left[i\left(n+\frac{1}{2}\right)\omega t\right], \tag{54}$$

which is exactly the same result as (50) (apart from an irrelevant complex conjugate) if one uses the properties of the Hermite polynomials (GRADSHTEYN & RIZHIK, 2000).

The transformation used in textbooks to introduce second quantization (path (d) in Figure 1) does not present the factors μ and ν. Strictly speaking, this transformation does not correspond to a classical canonical transformation. However, the μ and ν factors in (21) are clearly irrelevant, since they disappear with normalization in (54), and this is why their absence does not introduce any problem.

The representation in (52) is generally called *the number representation* because, if we use the transformed Hamiltonian in (22) expressed in terms of Q and P, we get

$$\hbar\omega\left(Q\frac{\partial}{\partial Q} + \frac{1}{2}\right)Q^n = \left(n+\frac{1}{2}\right)\hbar\omega Q^n, \tag{55}$$

n being called the *occupation number*.

V. Quantization of a Fermionic System

In reference (OLAVO & FIGUEIREDO, 1999), Olavo and Figueiredo proposed a phase-space representation for half-integral spins. They began from a classical material model for particles presenting half-integral spins which lead to the functions

$$S_1 = \frac{1}{2\omega}\left(\frac{p_x p_y}{m} + m\omega^2 xy\right), \tag{56}$$

$$S_2 = \frac{1}{4\omega}\left[m\omega^2(x^2-y^2) + \frac{1}{m}(p_x^2 - p_y^2)\right], \tag{57}$$

$$S_3 = \frac{1}{2}\left(xp_y - yp_x\right), \tag{58}$$

and

$$S_0 = \frac{1}{2\omega}\left[\frac{1}{m}(p_x^2+p_y^2) + m\omega^2(x^2+y^2)\right], \tag{59}$$

such that

$$S^2 = S_1^2 + S_2^2 + S_3^2 = \frac{1}{4}S_0^2, \tag{60}$$

and $\{S^2, S_j\} = 0$, for $j = 0,1,2,3$, where $\{*,*\}$ represents Poisson brackets.

In the quantization procedure, S^2 and S_3 were written in operator form. They were such that $\left[\hat{S}^2, \hat{S}_j\right] = 0$, for $j = 0,1,2,3$, where $[*,*]$ represents the Dirac commutator. The half integral spin *eigenfunctions* were found as the functions that makes these two operators diagonal (OLAVO & FIGUEIREDO, 1999).

From equation (60) we see that making diagonal the operator $\hat{S}_0$ is the same as making $\hat{S}^2$ diagonal. Indeed, it was shown in reference (OLAVO & FIGUEIREDO, 1999) that if we write

$$\hat{S}_0\psi = \hbar\lambda\psi, \tag{61}$$

then we get the operator $\hat{S}^2_\hbar = \hat{S}^2_0 - \hbar^2/4$ such that

$$\hat{S}^2_\hbar\psi = \hbar^2\left(\frac{\lambda-1}{2}\right)\left(\frac{\lambda+1}{2}\right)\psi, \tag{62}$$

or, with $N = \lambda - 1$,

$$\hat{S}^2_\hbar\psi = \hbar^2\left(\frac{N}{2}\right)\left(\frac{N}{2}+1\right)\psi. \tag{63}$$

We can now prove that this approach, with the phase-space functions (56)-(59) gives, in the second quantization formalism, exactly the same results of Schwinger's representation for half-integral spin systems, giving soundness to the phase-space representation of half-integral spins.

To apply the same quantization procedure used before we transform the extended canonical transformations (14) through the scale transformation

$$\mathcal{Q} = \mu' Q \quad \text{and} \quad \mathcal{P} = \nu' P, \tag{64}$$

where

$$\mu' = \frac{1}{\mu} \quad \text{and} \quad \nu' = \frac{1}{\nu} \Rightarrow \mu'\nu' = -i \neq 1, \tag{65}$$

as it should be for an extended canonical transformation (GOLDSTEIN, 2011).

From now on Q and P shall be understood under the canonical transformation (64), but we use the same notation as before for convenience. In this context we shall be interested in the following canonical transformations:

$$Q_1 = \left(\sqrt{\frac{m\omega}{2}}x + i\sqrt{\frac{1}{2m\omega}}p_x\right), \tag{66}$$

$$Q_2 = \left(\sqrt{\frac{m\omega}{2}}y + i\sqrt{\frac{1}{2m\omega}}p_y\right), \tag{67}$$

$$P_1 = \left(\sqrt{\frac{m\omega}{2}}x - i\sqrt{\frac{1}{2m\omega}}p_x\right), \tag{68}$$

and

$$P_2 = \left(\sqrt{\frac{m\omega}{2}}y - i\sqrt{\frac{1}{2m\omega}}p_y\right), \tag{69}$$

With these transformations, our functions S_i become

$$S_0' = (Q_1P_1 + Q_2P_2), \tag{70}$$

$$S_1' = \frac{1}{2}\left(Q_1 P_2 + Q_2 P_1\right), \tag{71}$$

$$S_2' = \frac{1}{2}\left(Q_1 P_1 - Q_2 P_2\right) \tag{72}$$

and

$$S_3' = \frac{1}{2i}\left(Q_2 P_1 - Q_1 P_2\right), \tag{73}$$

where the prime indicates that the functions are now represented in the transformed coordinate system.

We choose to diagonalize S_0' and S_3' at the same time. First of all, since we showed in Sec. 2 and 3 that both representations to obtain the USE and SQSE are equivalent, writing $P_j = -i\hbar\partial/\partial Q_j$ with $j = 1,2$ and following the same quantization procedure as the one we used to get equation (42), we obtain the operators

$$\hat{S}_0' = -i\hbar\left(Q_1 \frac{\partial}{\partial Q_1} + Q_2 \frac{\partial}{\partial Q_2}\right) \tag{74}$$

and

$$\hat{S}_3' = -\frac{\hbar}{2}\left(Q_2 \frac{\partial}{\partial Q_1} - Q_1 \frac{\partial}{\partial Q_2}\right). \tag{75}$$

In a similar definition, we shall use the operators

$$a_j^\dagger = e^{i\omega t} Q_j \quad \text{and} \quad a_j = e^{-i\omega t}\frac{\partial}{\partial Q_j}, \tag{76}$$

with $j = 1,2$, and the corresponding commutation relations (43) so that

$$\hat{S}_0' = -i\hbar\left(a_1^\dagger a_1 + a_2^\dagger a_2\right) \tag{77}$$

and

$$\hat{S}_3' = -\frac{\hbar}{2}\left(a_2^\dagger a_1 - a_1^\dagger a_2\right). \tag{78}$$

Instead of working with the operator $\hat{S}_0'$ we can work with a new one (as we did in equation (62)) given by

$$\hat{N} = i\hbar\left(a_1^\dagger a_1 + a_2^\dagger a_2\right) \tag{79}$$

with eigenvalues (see equation (61))

$$N = \lambda - 1. \tag{80}$$

In terms of the total 'angular momentum' we have a similar problem, i.e., we need to make diagonal the operators $\hat{S}_3'$ and

$$\hat{S}'^2 = \hbar\left(\frac{\hat{N}}{2}\right)\left(\frac{\hat{N}}{2} + 1\right). \tag{81}$$

The not normalized eigenvectors are given by the tensor product in Dirac representation

$$|\phi_{n_1}\phi_{n_2}\rangle = (a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}e^{i\omega t}\left|\phi_0^1\phi_0^2\right\rangle , \tag{82}$$

which are precisely the ones obtained in the literature (BAYM, 1969). We also note that the operators $\hat{S}_i'$ given by

$$S_1' = -\frac{i\hbar}{2}\left(a_1^\dagger a_2 + a_2^\dagger a_1\right) , \tag{83}$$

$$S_2' = -\frac{i\hbar}{2}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right) , \tag{84}$$

and

$$S_3' = -\frac{i\hbar}{2i}\left(a_2^\dagger a_1 - a_1^\dagger a_2\right) , \tag{85}$$

can be rewritten multiplying the expressions by $\mu\nu = i$ from the scale transformation (64) back to the previous coordinates (Q, P), as

$$S_1' = \frac{\hbar}{2}\left(a_1^\dagger a_2 + a_2^\dagger a_1\right) , \tag{86}$$

$$S_2' = \frac{\hbar}{2}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right) , \tag{87}$$

and

$$S_3' = \frac{\hbar}{2i}\left(a_2^\dagger a_1 - a_1^\dagger a_2\right) , \tag{88}$$

matching the ones in the Schwinger representation of angular momentum (DING & XU, 2014; SCHWINGER, 1952).

V.1. The classical representation of half-integral spin

The presentation of the quantum mechanical approach to half-integral spin systems generally introduce a disruption in the sequence of teaching of Quantum Mechanics. This teaching is generally done in terms of the usual Schrödinger equation, when step-like potentials, oscillators and atoms are considered. However, when one comes at the theme of half-integral spins, the process of teaching suddenly changes to a matrix-like description.

The general assumption behind this sudden change is that the half-integral spin systems do not have a *classical analog*, differently from oscillators, atoms and molecules. Thus, having a *classical analog* is here assumed as having a classical Hamiltonian (as the systems already mentioned), such that a Schrödinger equation based on $\hat{q}, \hat{p}$ operators obtains in the usual quantization process. This is the *meaning* of "having a classical analog".

However, what the previous developments show is that *there is* such a classical representation and, in fact, it was always at hand after Schwinger had introduced his representation. This seems obvious when one thinks of the creation and annihilation operators as position and momentum operators, however in a distinct representation, which are the result of canonical transformation on some Hamiltonian function.

What the previous section shows is the explicit (q, p) representation for these half-integral spin systems. It also shows that those working within the Schwinger representation are, indeed, working with a classical representation of the spin, on which it was performed quantization, although not necessarily knowing that.

VI. Conclusion

The Schrödinger equation may come in different notations. The second quantization notation is just only one of them. This notation comes as a natural result of a canonical transformation on the original classical phase space. Thus, passing from the usual (q, p) representation to the second quantized $(a^{\dagger}, a)$ representation does not represent any disruption, or need to avoid the use of Schrödinger equation based on differential operators. On the contrary, despite some technical advantages that it may bring about, the second quantization formalism simply hides a differential Schrödinger equation.

This means that *any* system that can be written in terms of such operators must have a classical analog, as was shown for the half-integral spin systems, although the explicit classical representation may be difficult to grasp.

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