REVISTA INTERDISCIPLINAR DE PESQUISA EM ENGENHARIA

CILAMCE 2016

XXXVII IBERIAN LATIN AMERICAN CONGRESS
ON COMPUTATIONAL METHODS IN ENGINEERING
BRASÍLIA - DF - BRAZIL

DYNAMIC ANALYSIS OF ELASTICALLY SUPPORTED TIMOSHENKO BEAM

Anderson Soares da Costa Azevêdo

Ana Carolina Azevedo Vasconcelos

Simone dos Santos Hoefel

anderson4r2@hotmail.com

carol-2506@hotmail.com

simone.santos@ufpi.edu.br

Universidade Federal do Piauí

Campus Universitário Ministro Petrônio Portella – Bairro Ininga, CEP: 64049-550, Teresina, Piauí, Brazil

Abstract. *The flexible beams carrying attachments and ends elastically restrained against rotational and translation inertia often appear in engineering structures, modal analysis of those structures is important and necessary in structural design. In case of structure with large aspect ratio of height and length the Timoshenko beam theory (TBT) is used, instead of the Euler-Bernoulli theory (EBT), since it takes both shear and rotary inertia into account. Shear effect is extremely large in higher vibration modes due to reduced mode half wave length. In this paper, the full development and analysis of TBT for transversely vibrations uniform beam are presented for elastically supported ends. A two-node beam element with two degree of freedom per node is obtained based upon Hamilton's principle. The influence of stiffnesses of the supports on the free vibration characteristics is investigated. For this purpose, the eigenvalues of the Timoshenko beam are calculated for various rigidity values of translational and rotational springs. The results obtained are discussed and compared with results obtained by other researchers.*

Keywords: *Finite element method, Timoshenko beam theory, Dynamic analysis*

1 INTRODUCTION

The dynamic analysis of beams with ends elastically restrained against rotation and translation or with ends carrying concentrated masses or rotational inertia is a classical structural problem. Various researchers have approached the analysis assuming that the beam is sufficiently slender to be considered as an Euler-Bernoulli beam, and trying to analytically solve the resulting fourth-order differential equation with variable coefficients. In exact method, difficulties arises in solving roots of the characteristic equation, except for very simple boundary conditions, that one has to go for numerical solution and in determining the normal modes of the system.

In 1972, Chun (1972) studied the free vibration of an Euler-Bernoulli beam hinged at one end by a rotational spring and with the other end free. Lee (1973) and Grant (1975) obtained the frequency equation for a uniform beam hinged at one end by a rotational spring and having a mass attached at the other end. MacBain and Genin (1973) investigated the effect of rotational and translational support flexibilities on the fundamental frequency of an almost-clamped-clamped Euler-Bernoulli beam. Goel (1976) investigated the vibration problem of a beam with an arbitrarily placed concentrated mass and elastically restrained against rotation at either end by using Laplace transforms. The deflection of a rotationally restrained beam was obtained by Nassar and Horton (1976) by successively integrating the differential equation of equilibrium and satisfying the appropriate boundary conditions, while Maurizi et al. (1976) studied the free vibration of a beam hinged at one end by a rotational spring and subjected to the restraining action of a translational spring at the other end. The exact solutions for the problems governed by a general self-adjoint fourth-order nonhomogeneous ordinary differential equation with arbitrarily polynomial varying coefficients and general elastic boundary conditions are derived in Greens function form was presented by Lee and Kuo (1992).

Free vibration frequencies of cantilever beam with variable cross-section and constraining spring was examined by Craver and Jampala (1993). Grossi and Arenas (1996), employed both the classical Rayleigh-Ritz method and the optimized Rayleigh-Schmidt method to find the frequencies with varying width and height. A lot of numerical results were given, for various non-classical boundary conditions. De Rosa and Auciello (1996) gave the exact free frequencies of a beam with linearly varying cross-section, in the presence of generic non-classical boundary conditions, so that all the usual boundary conditions can be treated as particular cases. Wang et al. (2007) utilized the Fourier series to investigate the dynamic analysis of beams having arbitrary boundary conditions. Yeih et al. (1999) employed a dual multiple reciprocity method (MRM) to determine the natural frequencies and natural modes of an Euler-Bernoulli beam. Liu and Gurram (2009) used Hes variational iteration method to calculate the natural frequencies and mode shapes of an Euler-Bernoulli beam under various supporting conditions.

Fourier series, also was used by Kim and Kim (2001) to obtain frequency expressions for uniform beams with generally restrained boundary conditions. The transverse vibration of uniform Euler-Bernoulli beams under linearly varying axial force was presented by Naguleswaran (2004). Kocaturk and Simsek (2005) used the Lagrange equations with the trial functions in the power series form denoting the deflection and the rotation of the cross-section of the beam for analyses of free vibration of elastically supported Timoshenko beams. Wang et al. (2007) studied the natural frequencies and mode shapes of a uniform Timoshenko beam carrying multiple intermediate spring-mass systems using an exact as well as a numerical assembly method.

Numerical results reveal that the effect of the shear deformation and rotary inertia joint terms on the lowest five natural frequencies of the combined vibrating system is somehow complicated. Prasad and Krishnamurthy (1973) and Rao and Raju (1974) have applied the Galerkin finite element method to vibration problems of beams. These studies have shown that the method is reliable and very accurate. The problem of free vibration of Timoshenko beams with elastically supported ends is solved by Abbas (1984), using the unique finite element model developed by Abbas (1979) which can satisfy all the geometric and natural boundary conditions of an elastically restrained Timoshenko beam. The effects of the translational and rotational support flexibilities on the natural frequencies of free vibrations of Timoshenko beams with non-idealized end conditions are investigated in detail. Undamped natural frequencies and the corresponding mode of vibration of a two-dimensional Timoshenko beam-column with generalized support conditions (i.e., with semirigid flexural restraints and lateral bracings as well as lumped masses at both ends) and subjected to a constant axial load along its span was derived, in a classic manner, by Aristizaba-Ochoa (2007). The model includes the simultaneous effects (or couplings) of bending and shear deformations, translational and rotational inertias of all masses considered. Hernandez et. al (2008) analyzed a mixed finite element method for computing the vibration modes of a Timoshenko curved rod with arbitrary geometry.

In this paper, free vibration of elastically supported beams is investigated based on TBT. The motions equation are derived from Euler-Lagrange equation. The stiffness and mass matrices for a two-node beam element with two degree of freedom per node is obtained based upon Hamilton's principle. Cubic and quadratic Lagragian polynomials are made interdependent by requiring them to satisfy both of the homogeneous differential equations associated with Timoshenko's beam theory. The influence of stiffness of the supports on the free vibration characteristics of TBT is investigated. For this purpose, the eigenvalues of the Timoshenko beam are calculated for various rigidity values of translational and rotational springs. The results obtained by Finite Element Method (FEM) are discussed and compared whit analytical solutions.

2 REVIEW OF BEAM EQUATION

The governing coupled differential equations for transverse vibrations of Timoshenko beams are (Timoshenko, 1921):

$$EI\frac{\partial^4 v(x,t)}{\partial x^4}+\rho A\frac{\partial^2 v(x,t)}{\partial t^2}-\frac{\rho EI}{KG}\frac{\partial^4 v(x,t)}{\partial x^2 t^2}-\rho I\frac{\partial^4 v(x,t)}{\partial x^2 t^2}+\frac{\rho^2 I}{KG}\frac{\partial^4 v(x,t)}{\partial t^4}=0, \quad (1)$$

$$EI\frac{\partial^4 \psi(x,t)}{\partial x^4}+\rho A\frac{\partial^2 \psi(x,t)}{\partial t^2}-\frac{\rho EI}{KG}\frac{\partial^4 \psi(x,t)}{\partial x^2 t^2}-\rho I\frac{\partial^4 \psi(x,t)}{\partial x^2 t^2}+\frac{\rho^2 I}{KG}\frac{\partial^4 \psi(x,t)}{\partial t^4}=0, \quad (2)$$

in which E is the modulus of elasticity, I, the moment of inertia of cross section, K, the shear coefficient, A, the cross-sectional area, G, the modulus of rigidity, ρ the mass per unit volume, $v(x,t)$, the transverse deflection, and $\psi(x,t)$ the bending slope. Assume that the beam is excited harmonically with a frequency ω and

$$v(x,t)=V(x)e^{jft}, \qquad \psi(x,t)=\Psi(x)e^{jft}, \qquad \text{and} \qquad \xi=x/L, \quad (3)$$

where $j=\sqrt{-1}$, ξ is the non-dimensional length of the beam, $V(x)$ is normal function of $v(x)$, $\Psi(x)$ is normal function of ψ(x), and L, the length of the beam. Substituting the above relations into Eq. (1) through Eq. (2) and omitting the common term e^{jft}, the following equations are obtained (Soares and Hoefel, 2015):

$$\frac{d^4V}{d\xi^4} + b^2(r^2+s^2)\frac{d^2V}{d\xi^2} - b^2(1-b^2r^2s^2)V = 0, \tag{4}$$

$$\frac{d^4\Psi}{d\xi^4} + b^2(r^2+s^2)\frac{d^2\Psi}{d\xi^2} - b^2(1-b^2r^2s^2)\Psi = 0 \tag{5}$$

with:

$$b^2 = \frac{\rho A L^4}{EI} f^2 \quad and \quad \omega = 2\pi f, \tag{6}$$

where ω is angular frequency, f, the natural frequency, and r and s are coefficients related with the effect of rotatory inertia and shear deformation given by:

$$r^2 = \frac{I}{AL^2} \qquad \text{and} \qquad s^2 = \frac{EI}{KAGL^2}, \tag{7}$$

We must consider two cases when obtaining Timoshenko beam model spatial solution (Soares and Hoefel, 2016). In the first case, assume:

$$\sqrt{(r^2-s^2)^2 + 4/b^2} > (r^2+s^2) \quad \text{which leads to} \quad b < \frac{1}{(r\,s)}, \tag{8}$$

while in the second:

$$\sqrt{(r^2-s^2)^2 + 4/b^2} < (r^2+s^2) \quad \text{which leads to} \quad b > \frac{1}{(r\,s)}. \tag{9}$$

Substituting $b = 1/(rs)$ in b relation presented on Eq. (6), we have the critical frequency expressed as:

$$\omega_{crit} = \sqrt{\frac{KGA}{\rho I}} \quad \text{or} \quad f_{crit} = \frac{\sqrt{KGA/\rho I}}{2\pi}. \tag{10}$$

We call this cutoff value $b_{crit} = 1/(rs)$. When $b < b_{crit}$ the solution of Eqs. (1) and (2) can be expressed respectively, in trigonometric and hyperbolic functions:

$$V(\xi) = C_1 cosh(\alpha_1\xi) + C_2 sinh(\alpha_1\xi) + C_3 cos(\beta\xi) + C_4 sin(\beta\xi), \tag{11}$$

$$\Psi(\xi) = C_1' sinh(\alpha_1\xi) + C_2' cosh(\alpha_1\xi) + C_3' sin(\beta\xi) + C_4' cos(\beta\xi) \tag{12}$$

with

$$\alpha_1 = \frac{b}{\sqrt{2}}\left[-(r^2+s^2) + \sqrt{(r^2-s^2)^2 + \frac{4}{b^2}}\right]^{1/2} \tag{13}$$

and

$$\beta = \frac{b}{\sqrt{2}}\left[(r^2+s^2) + \sqrt{(r^2-s^2)^2 + \frac{4}{b^2}}\right]^{1/2}. \tag{14}$$

When $b > b_{crit}$ the solution $V(\xi)$ and $\Psi(\xi)$ can be expressed only in trigonometric functions:

$$V(\xi) = \overline{C}_1 cos(\alpha_2\xi) + \overline{C}_2 sin(\alpha_2\xi) + \overline{C}_3 cos(\beta\xi) + \overline{C}_4 sin(\beta\xi), \tag{15}$$

$$\Psi(\xi) = \overline{C}'_1 sin(\alpha_2\xi) + \overline{C}'_2 cos(\alpha_2\xi) + \overline{C}'_3 sin(\beta\xi) + \overline{C}'_4 cos(\beta\xi) \tag{16}$$

with

$$\alpha_{t2'} = j\frac{b}{\sqrt{2}}\left[(r^2+s^2) - \sqrt{(r^2-s^2)^2 + \frac{4}{b^2}}\right]^{1/2} = j\,\alpha_{t2} \tag{17}$$

and

$$\beta_t = \frac{b}{\sqrt{2}}\left[(r^2+s^2) + \sqrt{(r^2-s^2)^2 + \frac{4}{b^2}}\right]^{1/2}. \tag{18}$$

The relations between the coefficients in Eqs. (11) and (12), or Eqs. (15) and (16) can be found in Huang (1961). The application of boundary condition will result in a infinite series of natural frequencies f_i, each associated with a particular mode shape. The Table 1 presents some boundary conditions, where k_m and k_r are, respectively, the translational and rotational spring constant, $S_o = 1$ for right end and $S_o = -1$ for left end.

Table 1: Boundary Conditions

Boundary Condition	Shear Force	Moment	Total Slope	Deflection
Hinged	-	$\frac{\partial\Psi(\xi)}{\partial\xi} = 0$	-	$V(\xi) = 0$
Clamped	-	-	$\Psi(\xi) = 0$	$V(\xi) = 0$
Free	$\Psi(\xi) - \frac{1}{L}\frac{\partial V(\xi)}{\partial\xi} = 0$	$\frac{\partial\Psi(\xi)}{\partial\xi} = 0$	-	-
Sliding	$\Psi(\xi) - \frac{1}{L}\frac{\partial V(\xi)}{\partial\xi} = 0$	-	$\Psi(\xi) = 0$	-
Linear spring	$S_oKGA\left[\frac{\partial V(\xi)}{\partial\xi} - \Psi(\xi)\right] = k_m V(\xi)$	$\frac{\partial\Psi(\xi)}{\partial\xi} = 0$	-	-
Torsional spring	$\Psi(\xi) - \frac{1}{L}\frac{\partial V(\xi)}{\partial\xi} = 0$	$S_oEI\frac{\partial\Psi(\xi)}{\partial\xi} = k_r\Psi(\xi)$	-	-

3 FINITE ELEMENT FORMULATION

Considering a element as shown in Fig. 1, the generalized coordinates at each node are V, the total deflection, and Ψ, the total slope. This results in a element with four degrees of freedom thus enabling the expression for V and Ψ to contain two undetermined parameters each, which can be replaced by the four nodal coordinates.

Using the non-dimension coordinate (ξ) and element length l_e defined in Fig. 1, the displacement V and total slope Ψ can be written in matrix form as follows:

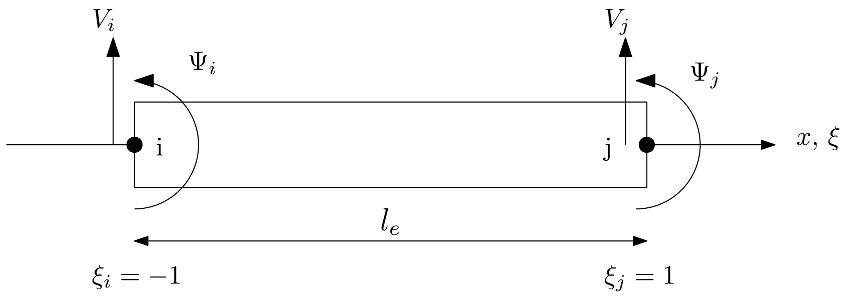

Figure 1: Beam element

$$V = [\,\mathbf{N}(\xi)\,]\{\,\mathbf{v}\,\}_e \quad \text{and} \quad \Psi = [\,\overline{\mathbf{N}}(\xi)\,]\{\,\mathbf{v}\,\}_e. \tag{19}$$

where

$$[\,\mathbf{N}(\xi)\,] = [\,N_1(\xi) \quad N_2(\xi) \quad N_3(\xi) \quad N_4(\xi)\,], \tag{20}$$
$$[\,\overline{\mathbf{N}}(\xi)\,] = [\,\overline{N}_1(\xi) \quad \overline{N}_2(\xi) \quad \overline{N}_3(\xi) \quad \overline{N}_4(\xi)\,]. \tag{21}$$

The displacements functions in Eqs. (20) and (21) can be expressed in terms of dimensionless parameters of rotatory r and shear s:

$$\mathbf{N_i}(\xi) = \frac{1}{4(3s^2+1)} \begin{bmatrix} 2(3s^2+1) - 3(2s^2+1)\xi + \xi^3 \\ a\,[3s^2+1-\xi-(3s^2+1)\xi^2+\xi^3] \\ 2(3s^2+1) + 3(2s^2+1)\xi - \xi^3 \\ a\,[-3s^2-1-\xi+(3s^2+1)\xi^2+\xi^3] \end{bmatrix}, \tag{22}$$

and

$$\overline{\mathbf{N}}_\mathbf{i}(\xi) = \frac{1}{4(3s^2+1)} \begin{bmatrix} a\,(3\xi^2-3) \\ -1-2(3s^2+1)\xi+6s^2+3\xi^2 \\ a\,(3-3\xi^2) \\ -1+2(3s^2+1)\xi+6s^2+3\xi^2 \end{bmatrix}, \tag{23}$$

where $a = l_e/2$.

Now, considering a linear spring connected to a beam, the potential and kinetic energy for an element length l_e of a uniform Timoshenko beam are given by:

$$\boldsymbol{U_e} = \frac{1}{2}\frac{EI}{a}\int_{-1}^{1}\left(\frac{\partial\Psi}{\partial\xi}\right)^2 d\xi + \frac{1}{2}\frac{EI}{a\,s^2}\int_{-1}^{1}\left(\frac{1}{a}\frac{\partial V}{\partial\xi} - \Psi\right)^2 d\xi + \frac{1}{2}k_m V^2 \tag{24}$$
$$\boldsymbol{T_e} = \frac{1}{2}\rho A a\int_{-1}^{1}\left(\frac{\partial V}{\partial t}\right)^2 d\xi + \frac{1}{2}r^2\rho A a^3\int_{-1}^{1}\left(\frac{\partial\Psi}{\partial t}\right)^2 d\xi, \tag{25}$$

Substituting the displacement expression (Eq. 19) into the potential energy (Eq. 24) gives:

$$
\begin{aligned}
\boldsymbol{U_e} \;=\; & \frac{1}{2}\,\{\mathbf{v}\}_e^T \left[\frac{EI}{a}\int_{-1}^{1}[\overline{\mathbf{N}}(\xi)']^T[\overline{\mathbf{N}}(\xi)']d\xi\right]\{\mathbf{v}\}_e \;+ \\
& \frac{1}{2}\,\{\mathbf{v}\}_e^T \left[\frac{EI}{a\,s^2}\int_{-1}^{1}[\mathbf{N}(\xi)' - \overline{\mathbf{N}}(\xi)]^T[\mathbf{N}(\xi)' - \overline{\mathbf{N}}(\xi)]d\xi\right]\{\mathbf{v}\}_e \;+ \\
& \frac{1}{2}\,\{\mathbf{v}\}_e^T \left[k_m[\mathbf{N}(\xi)]^T[\mathbf{N}(\xi)]\right]\{\mathbf{v}\}_e,
\end{aligned}
\tag{26}
$$

where $[\mathbf{N}(\xi)'] = [\partial \mathbf{N}(\xi)/\partial \xi]$. Therefore, the element stiffness matrix is given by:

$$
\begin{aligned}
[\boldsymbol{k_e}] \;=\; & \left[\frac{EI}{a}\int_{-1}^{1}[\overline{\mathbf{N}}(\xi)']^T[\overline{\mathbf{N}}(\xi)']d\xi \;+\right. \\
& \left.\frac{EI}{a\,s^2}\int_{-1}^{1}[\mathbf{N}(\xi)' - \overline{\mathbf{N}}(\xi)]^T[\mathbf{N}(\xi)' - \overline{\mathbf{N}}(\xi)]d\xi + k_m[\mathbf{N}(\xi)]^T[\mathbf{N}(\xi)]\right].
\end{aligned}
\tag{27}
$$

Substituting the displacement expression (Eq. 19) into the kinetic energy (Eq. 25) gives:

$$
\boldsymbol{T_e} = \frac{1}{2}\,\{\dot{\mathbf{v}}\}_e^T \left[\rho A a\int_{-1}^{1}[\mathbf{N}(\xi)]^T[\mathbf{N}(\xi)]d\xi + r^2\rho A a^3\int_{-1}^{1}[\overline{\mathbf{N}}(\xi)]^T[\overline{\mathbf{N}}(\xi)]d\xi\right]\{\dot{\mathbf{v}}\}_e,
\tag{28}
$$

The element mass matrix is given by:

$$
[\boldsymbol{m_e}] = \left[\rho A a\int_{-1}^{1}[\mathbf{N}(\xi)]^T[\mathbf{N}(\xi)]d\xi + r^2\rho A a^3\int_{-1}^{1}[\overline{\mathbf{N}}(\xi)]^T[\overline{\mathbf{N}}(\xi)]d\xi\right].
\tag{29}
$$

Torsional spring, whose ends are rotated in angular deflection, is presented similarly to linear spring, just adding the potential energy of the torsional spring to strain energy. So,

$$
\boldsymbol{U_e} = \frac{1}{2}\frac{EI}{a}\int_{-1}^{1}\left(\frac{\partial\Psi}{\partial\xi}\right)^2 d\xi + \frac{1}{2}\frac{EI}{a\,s^2}\int_{-1}^{1}\left(\frac{1}{a}\frac{\partial V}{\partial\xi} - \Psi\right)^2 d\xi + \frac{1}{2}k_r\Psi^2.
\tag{30}
$$

The kinetic energy of the element continues unchanged. Developing the Eq. (30) yields:

$$
\begin{aligned}
\boldsymbol{U_e} \;=\; & \frac{1}{2}\,\{\mathbf{v}\}_e^T \left[\frac{EI}{a}\int_{-1}^{1}[\overline{\mathbf{N}}(\xi)']^T[\overline{\mathbf{N}}(\xi)']d\xi\right]\{\mathbf{v}\}_e \;+ \\
& \frac{1}{2}\,\{\mathbf{v}\}_e^T \left[\frac{1}{a\,s^2}\int_{-1}^{1}[\mathbf{N}(\xi)' - \overline{\mathbf{N}}(\xi)]^T[\mathbf{N}(\xi)' - \overline{\mathbf{N}}(\xi)]d\xi\right]\{\mathbf{v}\}_e + \\
& \frac{1}{2}\,\{\mathbf{v}\}_e^T \left[k_r[\overline{\mathbf{N}}(\xi)]^T[\overline{\mathbf{N}}(\xi)]\right]\{\mathbf{v}\}_e,
\end{aligned}
\tag{31}
$$

The connection of more torsional or linear springs will add in the Eq. (30) and Eq. (24) more terms of potential energy of each spring. The torsional and linear springs are associated to the degree of freedom of slope and not to degree of freedom of vertical displacement.

Potential energy for Timoshenko beam including both translation and torsional springs is given by:

$$\begin{aligned}\boldsymbol{U_e} &= \frac{1}{2}\{\mathbf{v}\}_e^T\left[\frac{EI}{L}\int_{-1}^{1}[\overline{\mathbf{N}}(\xi)']^T[\overline{\mathbf{N}}(\xi)']d\xi\right]\{\mathbf{v}\}_e + \\ & \frac{1}{2}\{\mathbf{v}\}_e^T\left[\frac{1}{a\,s^2}\int_{-1}^{1}[\mathbf{N}(\xi)'-\overline{\mathbf{N}}(\xi)]^T[\mathbf{N}(\xi)'-\overline{\mathbf{N}}(\xi)]d\xi\right]\{\mathbf{v}\}_e + \\ & \frac{1}{2}\{\mathbf{v}\}_e^T\left[k_m[\mathbf{N}(\xi)]^T[\mathbf{N}(\xi)]\right]\{\mathbf{v}\}_e + \{\mathbf{v}\}_e^T\left[k_r[\overline{\mathbf{N}}(\xi)]^T[\overline{\mathbf{N}}(\xi)]\right]\{\mathbf{v}\}_e. \end{aligned} \tag{32}$$

4 Numerical Results

This section presents numerical examples for Timoshenko beams subjected to various translational and rotational stiffness coefficients. Consider a straight uniform Timoshenko beam of length L, $K = 5/6$ and Poisson's ratio $\nu = 0.3$. The beam is elastically restrained against translational and rotational at either end as shown in the Fig. (2), where k_m and k_r are the translational and rotational spring constant.

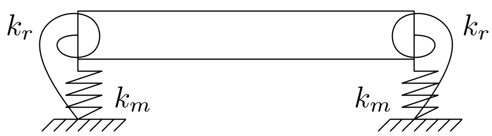

Figure 2: Timoshenko beam restricted with translational and rotational springs at both ends

In order to investigate the influence of stiffness of the supports on the free vibration characteristics of Timoshenko beams, the eigenvalues are calculated for a set of h/L ratios. The results obtained by FEM (discretizations with 30 and 70 elements) are compared with classical theory (C T) presented by Kocaturk and Simsek (2005). We observed the possibility to simulate infinite support stiffness by setting translational and rotational stiffness coefficients equal to 1×10^8. Figure 3 presents the four classical boundary conditions obtained by setting k_m and k_r to zero or 1×10^8.

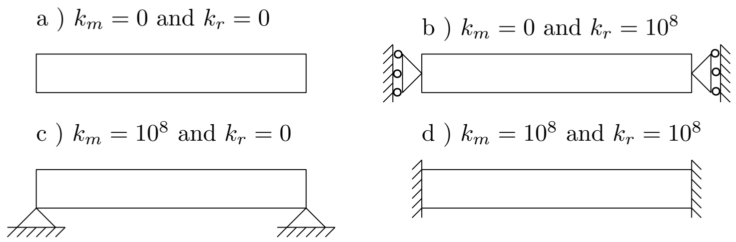

Figure 3: a) Free-free; b) Sliding-sliding; c) Hinged-hinged; d) Clamped-clamped beams

4.1 Hinged-hinged beam

Table 2 presents the first six eigenvalues for hinged-hinged ($k_m = 1 \times 10^8\, N/m$, $k_r = 0\, Nm/rad$). Notice that for higher modes, error decreases when the number of elements is increased. It is observed that the difference in the frequencies of EBT and TBT beams become significant with increase of the mode numbers and h/L ratio. However, the dimensionless

Table 2: Eigenvalues of the hinged-hinged Timoshenko beam

$h/L = 0.005$						
Eigenvalues	C T	FEM 30e	Error(%)	FEM 70e	Error(%)	EBT
b_1	3.14152	3.1415	6.3663e-04	3.1415	6.3663e-04	3.14159
b_2	6.28265	6.2827	7.9584e-04	6.2827	7.9584e-04	6.28319
b_3	9.42297	9.4230	3.1837e-04	9.4230	3.1837e-04	9.42478
b_4	12.56200	12.5623	2.3882e-03	12.5621	7.9605e-04	12.5664
b_5	15.69960	15.7002	3.8218e-03	15.6997	6.3696e-04	15.7080
b_6	18.83510	18.8366	7.9639e-03	18.8353	1.0618e-03	18.8496
$h/L = 0.020$						
Eigenvalues	C T	FEM 30e	Error(%)	FEM 70e	Error(%)	EBT
b_1	3.14053	3.1405	9.5525e-04	3.1405	9.5525e-04	3.14159
b_2	6.27470	6.2747	0.0000	6.2747	0.0000	6.28319
b_3	9.39630	9.3965	2.1285e-03	9.3963	0.0000	9.42478
b_4	12.49930	12.5003	8.0004e-03	12.4995	1.6001e-03	12.5664
b_5	15.57830	15.5810	1.7332e-02	15.5788	3.2096e-03	15.7080
b_6	18.62810	18.6345	3.4357e-02	18.6292	5.9051e-03	18.8496
$h/L = 0.050$						
Eigenvalues	C T	FEM 30e	Error(%)	FEM 70e	Error(%)	EBT
b_1	3.13499	3.1350	3.1898e-04	3.1350	3.1898e-04	3.14159
b_2	6.23136	6.2315	2.2467e-03	6.2314	6.4191e-04	6.28319
b_3	9.25536	9.2564	1.1237e-02	9.2556	2.5931e-03	9.42478
b_4	12.18130	12.1854	3.3658e-02	12.1821	6.5674e-03	12.5664
b_5	14.99260	15.0040	7.6038e-02	14.9947	1.4007e-02	15.7080
b_6	17.68090	17.7067	1.4592e-01	17.6857	2.7148e-02	18.8496

frequency parameter b_i of both theories are very close to each other for small values of h/L ratio.

Early researchers reported that Timoshenko beams presents changes on dynamic behavior for $b > b_{crit}$. On hinged-hinged beam is well known the existence of double eigenvalues with similar wave forms, but distinct because deformations due to shear and bending are on the same phase for $b < b_{crit}$ and out of phase for $b > b_{crit}$ (Downs, 1976). Beside that it is, observed that when $b = b_{crit}$ on hinged-hinged beam occurs a mode shape, called shear mode, with the motion as a shear oscillation without transverse deflection. For a rectangular cross-section beam, b_{crit} is written as a function of h/L ratio as follows:

$$b_{crit} = 6\left(\frac{L}{h}\right)^2 \sqrt{\frac{2K}{(1+\nu)}}. \qquad (33)$$

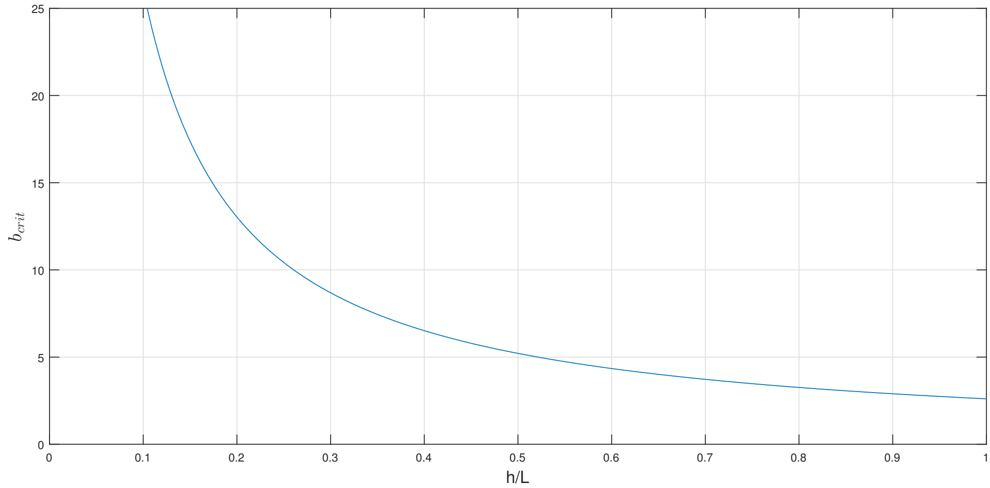

Figure 4: Critical frequency parameter

Figure 4 presents b_{crit} for various h/L ratios. We observed that b_{crit} become lower as h/L ratio increases. In order to show this behavior into lower modes, Fig. 5-a presents the first five mode shapes (solid-lines) for $b < b_{crit}$ and Fig. 5-b the first five mode shapes for $b > b_{crit}$ and shear mode (dashed-line) to a thicker beam of $h/L = 0.25$.

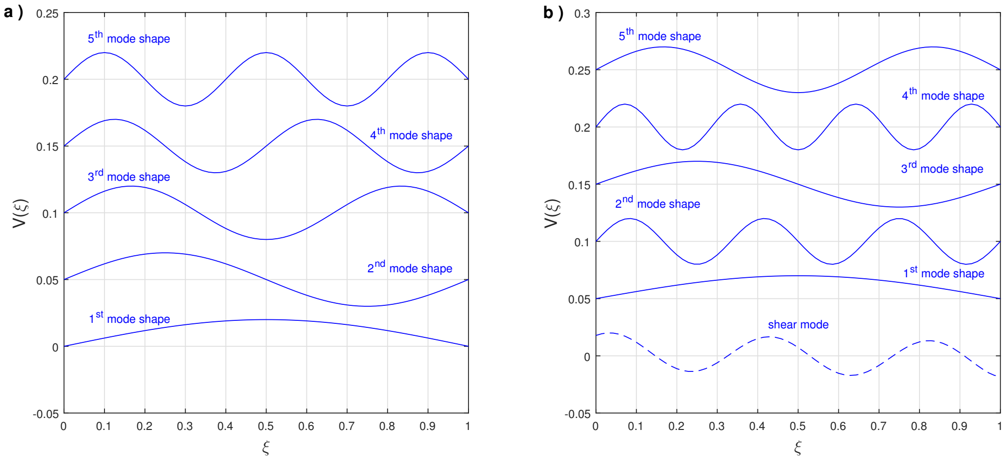

Figure 5: a) First five modes for $b < b_{crit}$; b) Shear mode and first five modes for $b > b_{crit}$ (FEM -70e)

Observe that on Fig. 5 shear mode does not fit the boundaries conditions as reported by Levison and Cooke (1982). The peak amplitude of the transverse deflection on each mode shape for all examples has been normalized to $0.02\,m$.

4.2 Clamped-clamped beam

Table 3 presents the first six eigenvalues for clamped-clamped ($k_m = 1 \times 10^8\ N/m$, $k_r = 1 \times 10^8\ Nm/rad$). The accuracy of presented finite element is also confirmed for clamped-clamped beam.We observed that the difference in the frequencies of EBT and TBT beams be-

Table 3: Eigenvalues of the clamped-clamped Timoshenko beam

$h/L = 0.005$						
Eigenvalues	CT	FEM 30e	Error(%)	FEM 70e	Error(%)	EBT
b_1	4.72962	4.7296	4.2287e-04	4.7314	3.7635e-02	4.73004
b_2	7.85161	7.8516	1.2736e-04	7.8499	2.1779e-02	7.85320
b_3	10.99160	10.9918	1.8196e-03	10.9952	3.2752e-02	10.9956
b_4	14.12920	14.1297	3.5388e-03	14.1307	1.0616e-02	14.1372
b_5	17.26520	17.2660	4.6336e-03	17.2582	4.0544e-02	17.2788
b_6	20.39850	20.4006	1.0295e-02	20.3952	1.6178e-02	20.4204
$h/L = 0.020$						
Eigenvalues	CT	FEM 30e	Error(%)	FEM 70e	Error(%)	EBT
b_1	4.72348	4.7235	4.2342e-04	4.7285	1.0628e-01	4.73004
b_2	7.82816	7.8282	5.1098e-04	7.8321	5.0331e-02	7.85320
b_3	10.93400	10.9345	4.5729e-03	10.9357	1.5548e-02	10.9956
b_4	14.01530	14.0169	1.1416e-02	14.0150	2.1405e-03	14.1372
b_5	17.06760	17.0719	2.5194e-02	17.0667	5.2731e-03	17.2788
b_6	20.08450	20.0959	5.6760e-02	20.0862	8.4642e-03	20.4204
$h/L = 0.050$						
Eigenvalues	CT	FEM 30e	Error(%)	FEM 70e	Error(%)	EBT
b_1	4.68991	4.6899	2.1322e-04	4.6899	2.1322e-04	4.73004
b_2	7.70350	7.7039	5.1924e-03	7.7036	1.2981e-03	7.85320
b_3	10.64010	10.6422	1.9737e-02	10.6405	3.7594e-03	10.9956
b_4	13.46100	13.4678	5.0516e-02	13.4624	1.0400e-02	14.1372
b_5	16.15890	16.1756	1.0335e-01	16.1620	1.9184e-02	17.2788
b_6	18.73180	18.7662	1.8364e-01	18.7380	3.3099e-02	20.4204

come significant with increase of the mode numbers and h/L ratio, similarly to hinged-hinged example.

Smith (2008) observed that on clamped-clamped TBT beam occurs a interesting phenomena, for $b < b_{crit}$ the number of peaks in the deflection increases by unit with increase in mode number, and for $b > b_{crit}$, the number of peaks increases with each pair of modes. In order to show this behavior for lower modes, Fig. 6-a presents the first five mode shapes for $b < b_{crit}$ and Fig. 6-b for $b > b_{crit}$ to a thicker beam of $h/L = 0.25$.

5 CONCLUSIONS

In this paper was presented a brief review of Timoshenko beam theory, and developed a two-node beam element with two degree of freedom per node base upon Hamilton's principle. The influence of the supports stiffness was investigated on Timoshenko beams elastically

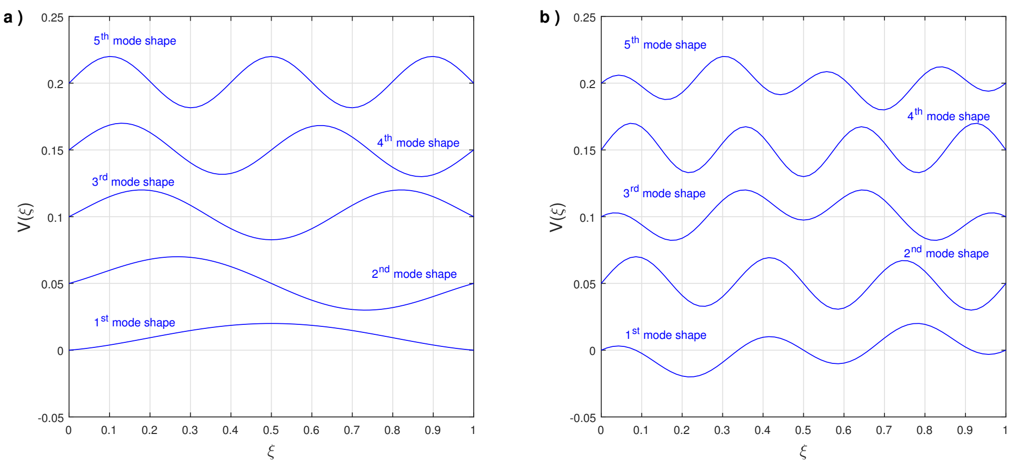

Figure 6: a) First five mode shapes for $b < b_{crit}$; b) First five mode shapes for $b > b_{crit}$ (FEM -70e)

restrained against rotational and translation inertia at both ends on the free vibration. We observed the possibility to obtain the four classical boundary conditions by simulating infinite or zero support stiffness on translational and rotational stiffness coefficients. Eigenvalues and mode shapes of hinged-hinged ($k_m = 1 \times 10^8\, N/m$ and $k_r = 0\, Nm/rad$) and clamped-clamped ($k_m = 1 \times 10^8\, N/m$ and $k_r = 1 \times 10^8\, Nm/rad$) beams were obtained in well agreement with other researchers results. Was observed that by manipulating the stiffness coefficients of both elastically supports for $b > b_{crit}$ is possible to obtain the double eigenvalues and shear mode well known for hinged-hinged beam, and the phenomena for clamped-clamped beam where the number of peaks in the deflection increases per unit with each pair of modes.

5.1 Permission

The authors are the only responsible for the printed material included in this paper.

REFERENCES

Abbas, B. A. H, 1979. Simple finite elements for dynamic analysis of thick pre-twisted blades. *Aeronautical Quaterly*, vol. 83, pp.450-453.

Abbas, B. A. H., 1984. Vibrations of beams with elastically restrained end. *Journal of Sound and Vibration*, vol. 97, pp. 541-548.

Aristizaba-Ochoa, D. J., 2007. Tension and Compression Stability and Second-Order Analyses of Three-Dimensional Multicolumn Systems: Effects of Shear Deformations. *Journal of Engineering Mechanics*, vol. 133, pp. 106-116.

Chun, K. R., 1972. Free vibration of a beam with one end spring-hinged and the other free. *Journal of Applied Mechanics*, vol. 39, pp. 1154-1155.

Craver, L. Jr., & Jampala, P., 1993. Transverse vibrations of a linearly tapered cantilever beam with constraining springs. *Journal of Sound and Vibration*, vol. 166, pp. 521-529.

De Rosa, M. A., & Auciello, N. M., 1996. Free vibrations of tapered beams with exible ends. *Computers & Structures*, vol. 60, pp.197-202.

Downs, B., 1976. Vibration of a uniform, simply supported Timoshenko beam without transverse deflection. *Journal of Applied Mechanics*, vol. 43, pp. 671-674.

Grant, D. A., 1975. Vibration frequencies for a uniform beam with one end elastically supported and carrying a mass at the other end. *Journal of Applied Mechanics*, vol. 42, pp. 878-880.

Goel, R. P., 1976. Free vibrations of a beam-mass system with elastically restrained ends. *Journal of Sound and Vibration*, vol. 47, pp. 9-14.

Grossi, R. O., & Arenas, B. del V., 1996. A variational approach to the vibration of tapered beams with elastically restrained ends. *Journal of Sound and Vibration*, vol. 195, pp. 507-511.

Hernandez, E., Otrola, E., Rodriguez, R., & Sahueza, F., 2008. Finite element aproximation of the vibration problem for a Timoshenko curved rod. *Revista de La Union Matemtica*, vol. 49, pp.15 - 28.

Huang, T. C., 1961. The effect of rotatory inertia and of shear deformation on the frequency and normal mode equations of uniform beams with simple end conditions. *Journal of Applied Mechanics*, vol. 28, pp. 579-584.

Kim, H. K., & Kim, M. S., 2001. Vibration of Beams with generally restrained boundary conditions using Fourier series. *Journal of Sound and Vibration*, vol. 245, pp. 771-784.

Kocaturk, T., & Simsek, M., 2005. Free vibration analysis of elastically supported Timoshenko beams. *Journal of Engineering and Natural Sciences*, vol.23 , pp. 79-93.

Lee, S. Y., & Kuo, Y. H., 1992. Exact Solutions for the Analysis of General Elastically Restrained Nonuniform Beams. *Journal of Applied Mechanics*, vol. 59, pp. 205-212.

Lee, T. W., 1973. Vibration frequencies for a uniform beam with one end spring-hinged and carrying a mass at the other free end. *Journal of Applied Mechanics*, vol. 40, pp. 813-815.

Levinson, M., & Cooke, D. W., 1982. On the two frequency spectra of Timoshenko beams. *Journal of Sound and Vibration*, vol. 84, pp. 319-326.

Liu, Y., & Gurram, C. S., 2009. The use of Hes variational iteration method for obtaining the free vibration of an EulerBernoulli beam. *athematical and Computer Modelling*, vol. 50, pp. 1545-1552.

MacBain, J. C., & Genin, 1973. Effect of support flexibility on the fundamental frequency of vibrating beams.*Journal of the Franklin Institute*, vol. 296, pp. 259-273.

Maurizi, M. J., & Rossr, R. E., Reyes, J. A., 1976. Vibration frequencies for a uniform beam with one end spring-hinged and subjected to a translational restraint at the other end. *Sound and Vibration*, vol. 48, pp. 565-568.

Naguleswaran, S., 2004. Transverse vibration of an uniform EulerBernoulli beam under linearly varying axial force. *Journal of Sound and Vibration*, vol. 275, pp. 47-57.

Nassar, E. M., & Horton, W. H., 1976. Static deflection of beams subjected to elastic rotational restraints. *American Institute of Aeronautics and Astronautics Journal*, vol. 14, pp. 122-123.

Prasad, K. S. R. K., & Krishnamurthy, A. V., 1973. Galerkin finite element method for vibration problems. *Institute of Aeronautics and Astronautics Journal*, vol. 11, pp. 544-546.

Rao, G. V., & Raju, K. K., 1974. A Galerkin finite element analysis of a uniform beam carrying a concentrated mass and rotary inertia with a spring hinge. *Journal of Sound and Vibration*, vol. 37, pp. 567-569.

Smith, R. W. M., 2008. Graphical representation of Timoshenko beam modes for clamped-clamped boundary conditions at high frequency: Beyond transverse deflection. *Wave Motion*, vol. 45, pp. 785-794.

Soares, A., & Hoefel, S. S., 2015. Modal analysis for free vibration of four beam theories. In *Proceedings of the 23rd International Congress of Mechanical Engineering - COBEM 2015*, Rio de Janeiro, Brazil.

Soares, A., & Hoefel, S. S., 2016. Analysis of rotatory inertia and shear deformation on transverse vibration of beams. In *Proceedings of the IX Congresso Nacional de Engenharia Mecânica - CONEM 2016*, Fortaleza, Brazil.

Timoshenko, S. P., 1921. On the correction for shear of the differential equation for transverse vibration of prismatic bars. *Philosophical Magazine*, vol. 41, pp. 744-746.

Wang, J. R., Liu, T.-L., & Chein, D-W, 2007. Free vibration analysis of a Timoshenko beam carrying multiple spring-mass systems with the effects of shear deformation and rotary inertia. *Structural Engineering & Mechanic*, vol. 26, pp. 1-14.

Yeih, W., Chen, J. T., & Chang, C. M., 1999. Applications of dual MRM for determining the natural frequencies and natural modes of an Euler-Bernoulli beam using the singular value decomposition method. *Engineering Analysis with Boundary Elements*, vol. 23, pp. 339-360.