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DYNAMIC BEHAVIOR OF FLEXIBLE PLATES SUPPORTED BY A TRANSVERSELY ISOTROPIC HALF-SPACE

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Abstract. *The analysis of flexible plates supported on single layered soil usually uses the Winkler model to simulate the displacements and soil pressure on the plate. However, this model presents serious limitations and it is not able to represent the lateral continuity of the soil. In this article a formulation for the analysis of flexible plates under harmonic dynamic loading, supported on the soil surface, modeled as homogeneous elastic, transversely isotropic half-space is shown. The plate is modeled by rectangular finite elements (FEM) and for the soil the indirect boundary element method (IBEM is used). Dynamic influence functions are used for the elastic transversely isotropic half-space. Therefore, only the interface soil-plate is discretized. The compatibility of the displacements between the plate elements and the soil elements is done in the central point of those elements. Hence, the discretization of the plate and the soil surface in contact are the same. Numerical results for rectangular plates supported by isotropic medium are compared with published results by other authors. The anisotropic effect of the soil in the system is also analyzed.*

Keywords: *Elastodynamics, Anisotropy, Boundary Element Method, Thin plates*

1 INTRODUCTION

The dynamic interaction between slabs and soil is of great interest for the foundation engineering; particularly for the design of machine foundations. In the literature is easy to find many articles concerning such foundations, but many of them analyze rigid rather than flexible plates. Furthermore, models using springs to simulate the soil, like Winkler model, are frequently seen. For dynamic analysis, the Winkler model has even more serious limitations because the propagation of the vibrations through the soil is not considered. An elastic continuum is a much more accurate model, although in this case the formulation becomes more complex and the computer implementation more expensive.

The analysis of dynamic interaction between flexible rectangular plates and the soil modeled as an elastic continuum has been presented by Savidis and Richter (1979), Iguchi and Luco (1981), Auersch (1996) and Quian et al. (1996). Those articles invariably employ the Finite Element Method (FEM) for the plate model and some form of Green's functions for the elastic medium bellow. The main differences between them are the medium geometry (half-space or layered medium) and the soil-plate compatibilization (forces and displacements) strategy.

For the case of anisotropic soils, only more recent works deal with the dynamic interactions between foundations and transversely isotropic (also known as cross-anisotropic) soils. Barros (2006) analyzed the dynamic response of rigid, cylindrical foundations embedded in a transversely isotropic, elastic half-space and Amiri-Hezaveh et al. (2013) analyzed the dynamic behavior of rectangular, rigid foundations on the surface of a transversely isotropic half-space. Also, Labaki (2012) and Labaki et al. (2014) presented the dynamic interaction between rigid and flexible circular plates and layered, transversely isotropic soils.

This work presents a method of analysis of flexible, rectangular plates on the surface of a transversely isotropic half-space, subjected to a time-harmonic dynamic, transverse load. The plate is modeled with four-node, rectangular finite elements and the soil response is obtained by the Indirect Boundary Element Method (IBEM) formulation which uses half-space Green's functions calculated for loads distributed along circular areas.

The soil-plate contact is divided in rectangular boundary elements which coincide with the plate elements and the soil reaction is assumed to be uniformly distributed along each element. By imposing soil-plate displacement compatibility at the center of each element, a system of equation is obtained. The solution of this system furnishes both the plate FEM nodes displacement and the soil interaction tractions.

2 PLATE MODEL

Based on Kirchhoff-Love's theory, the finite element method is used for the elements of the flexible plate. The element is rectangular and has 4 nodes, each node has 3 degrees of freedom: vertical displacement in z direction, rotation around the x axis and rotation around the y axis. Hence, the stiffness matrix and mass matrix have 12 degrees of freedom. The model uses n_e elements and n_n nodes, in which results a total of $3n_n$ degrees of freedom. Since the element has 12 degrees of freedom, it is selected a 12-term polynomial in x and y to represent vertical displacement over the entire surface of the plate.

$$w(x,y) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 + a_7 x^3 + a_8 x^2 y + a_9 xy^2 + a_{10} y^3 + a_{11} x^3 y + a_{12} xy^3. \quad (1)$$

The polynomial is an incomplete quartic and complete up to the third order. In the Pascal triangle is easy to see the two fourth-degree terms, which were included to keep the element geometrically isotropic. The stiffness matrix, the mass matrix and the vector of equivalent nodal forces for a constant pressure applied on the plate in the z direction are given by the usual process that derives them from energy principles. This element is called MZC element because it was originally developed by Melosh (1961), Zienkiewicz and Cheung (1964). An explicit form for the stiffness and mass matrices is given in Przemienieki (1968). After the assembly of the global stiffness and mass matrix of the plate, as well the vector of equivalent nodal forces and neglecting the structural damping, the equation obtained is:

$$([K] - \omega^2 [M])\{u\} = \{f\} - \{f_s\}. \quad (2)$$

where the matrix $[K]$ is the global stiffness matrix, the matrix $[M]$ is the global mass matrix, ω is the frequency of the dynamic load. The vector $\{u\}$ consists in the displacements and rotations at the nodes. The vector $\{f_s\}$ contains the equivalent nodal forces that result from the soil tractions applied to the plate. Vectors $\{u\}$ and $\{f_s\}$ are given by:

$$\{u\} = \begin{Bmatrix} w_1 \\ \varphi_{x1} \\ \varphi_{y1} \\ w_2 \\ \varphi_{x2} \\ \varphi_{y2} \\ \vdots \\ w_n \\ \varphi_{xn} \\ \varphi_{yn} \end{Bmatrix}_{3n_n \times 1}, \quad \{f\} = \begin{Bmatrix} f_{z1} \\ m_{x1} \\ m_{y1} \\ f_{z2} \\ m_{x2} \\ m_{y2} \\ \vdots \\ f_z \\ m_{xn} \\ m_{yn} \end{Bmatrix}_{3n_n \times 1}. \quad (3)$$

where x_i and y_i are coordinates of the node i.

The equivalent nodal forces that the half-space applies to the plate are given by:

$$\{f_s\} = [A]\{q\}. \quad (4)$$

where $\{q\}$ is the vector of applied loads by the half-space to the plate surface in each element. Assuming that these loads consist of normal tractions to the plate with uniform distribution on the element, the vector $\{q\}$ is given by:

$$\{q\} = \begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_{n_e} \end{Bmatrix}_{n_e \times 1}. \quad (5)$$

It should be noted that the soil traction component tangent to the soil-plate interface is not considered in this analysis. This assumption implies that the soil-plate interface has zero friction and the interface is free to slide.

The matrix $[A]$ is assembled by the columns, each column corresponding to an element and the row positions in that column corresponding to the element degrees of freedom. The values to be filled in those positions are the equivalent nodal forces due to an uniformly distributed load; the values in the remaining positions are zero. Then, the matrix takes the form:

$$[A] = \begin{Bmatrix} \dfrac{l_{x1}l_{y1}}{4} & 0 & \vdots & \vdots \\ \dfrac{l_{x1}l_{y1}^2}{24} & 0 & \vdots & \vdots \\ -\dfrac{l_{x1}^2 l_{y1}}{24} & 0 & \vdots & \vdots \\ 0 & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{Bmatrix}_{3n_n \times n_e} . \tag{6}$$

where l_{xi} and l_{yi} are the lengths of the element i.

The final form of the equation of the plate results:

$$([K] - \omega^2 [M])\{u\} + [A]\{q\} = \{f\} . \tag{7}$$

3 HALF-SPACE MODEL

The boundary element method has two varieties, called direct and indirect formulations. In this study, the half-space is represented by the indirect formulations. The contact area between the plate and the half-space is divided in rectangular areas according to the exact division of the plate elements.

The present work introduces a formulation for the boundary element method, which is based on Green’s functions (influence functions), which are the elastic medium response to applied loads. Influence functions for dynamic loads distributed along disks applied on a transversely isotropic half-space were derived by Rajapakse and Wang (1993). These loaded disks are used in place of rectangular elements, with an equivalent radius for each element given by:

$$r = \sqrt{\frac{l_x l_y}{\pi}} . \tag{8}$$

Transforming rectangular elements into equivalent circular elements may cause an error. Depending on the relation between the sides of the element, the error increases affecting the final result, but if it is chosen an adequate element with $L/B = 1.5$ or less, this error can be extremely low, without influencing the final results. The Table 1 below illustrates the difference of the displacements at the center of the loaded area, between a rectangular area and its equivalent circular area, where a static uniformly distributed load was considered. For those conditions an analytical solution is quite simple and easily found in the literature.

Table 1. Relation between circular plate and rectangular plate

L/B	R_e/B	I_p	I_p'	Difference
1	0.564	1.122	1.128	0.53%
1.5	0.691	1.358	1.382	1.77%
2.0	0.798	1.532	1.596	4.18%
3.0	0.977	1.783	1.954	9.59%

Under axisymmetric conditions, the stress-strain relationship for the transversely isotropic medium, is given by:

$$\sigma_{rr} = c_{11}\varepsilon_{rr} + c_{13}\varepsilon_{zz}. \quad (9)$$

$$\sigma_{zz} = c_{13}\varepsilon_{rr} + c_{33}\varepsilon_{zz}. \quad (10)$$

$$\sigma_{rz} = 2c_{44}\varepsilon_{rz}. \quad (11)$$

The isotropic case is a special one, with $c_{11} = c_{33} = \lambda + 2\mu$, $c_{13} = \lambda$ and $c_{44} = \mu$, where λ and μ are the Lamè's constants. The anisotropy of the material under axisymmetric loading may be expressed by two anisotropy indices n_1 and n_3 given by:

$$n_1 = c_{33} / c_{11}. \quad (12)$$

$$n_3 = (c_{11} - 2c_{44}) / c_{13}. \quad (13)$$

The displacements in the z direction $\{w_s\}$ in the center of each rectangular element due to a distributed vertical load are given by:

$$\{w_s\} = [U]\{q\}. \quad (14)$$

where:

$$\{w_s\} = \begin{Bmatrix} w_{s1} \\ w_{s2} \\ \vdots \\ w_{sne} \end{Bmatrix}_{n_e \times 1} . \quad (15)$$

and $[U]$ is the influence matrix. Each element U_{ij} of this matrix corresponds to the displacement in the center of the element i due to a distributed unitary load applied at the element j. The Green's functions for a load applied on the surface of a transversely isotropic half-space have the general form, in cylindrical coordinates (Rajapakse and Wang, 1993):

$$w(r) = \int_0^\infty w^*(\varsigma)\, p^*(\varsigma)\, d\varsigma . \quad (16)$$

where $w(r)$ is the vertical displacement on the half-space, w^* is a function of the elastic constants, the frequency and of the distance r from the load center to the observation point; and p^* is the Hankel transform of the applied load.

The displacements $\{w_s\}$ are related to the total nodal displacements and rotations of the corresponding plate element. Thus:

$$\{w_s\} = [D]\{u\}. \tag{17}$$

where $[D]$ is assembled by rows, with each row corresponding to an element and filled in the corresponding positions to the degrees of freedom of the element. The values are the nodal displacements and rotations equivalent to a vertical displacement in the center of the element.

$$[D] = \begin{bmatrix} \frac{1}{4} & \frac{l_{y1}}{16} & -\frac{l_{x1}}{16} & 0 & \cdots \\ 0 & 0 & 0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{n_e \times 3n_n}. \tag{18}$$

The equation for the half-space becomes:

$$[D]\{u\} - [U]\{q\} = 0. \tag{19}$$

4 GLOBAL SYSTEM OF EQUATIONS

Assuming the vertical displacements at the center of the rectangular elements on the half-space are equal to the vertical displacements at the center of the plate elements and the normal stress in the interface between the plate and the half-space can be approximated by the composition of uniform distributed load rectangles, a global system of equations can be assembled:

$$\begin{bmatrix} [K] & [A] \\ [D] & -[U] \end{bmatrix} \begin{Bmatrix} \{u\} \\ \{q\} \end{Bmatrix} = \begin{Bmatrix} \{f\} \\ \{0\} \end{Bmatrix}. \tag{20}$$

The solution of the global system of equations furnishes the displacements and rotations at the FEM model nodes and the soil-plate interface tractions. Figure 1 shows the connectivity between the plate elements and the soil elements.

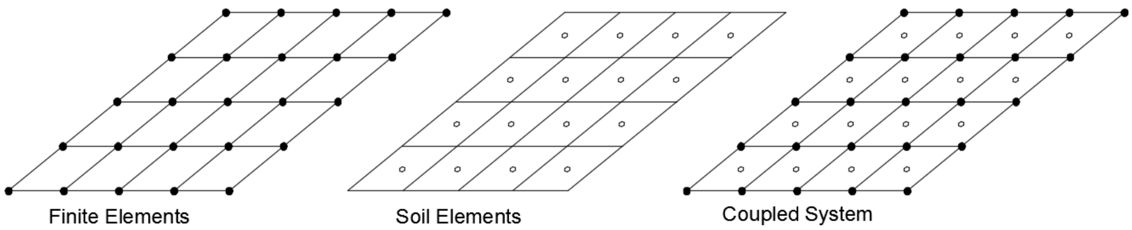

Figure 1. Finite elements and soil elements discretized and the coupled system

For the case of a perfectly rigid plate, the FEM model is not necessary. Displacement compatibility between plate and soil; along with plate equilibrium reduces Eqn. (20) to:

$$\begin{bmatrix} -\omega^2 m & \{S\} \\ \{1\} & -[U] \end{bmatrix} \begin{Bmatrix} w_r \\ \{q\} \end{Bmatrix} = \begin{Bmatrix} P \\ \{0\} \end{Bmatrix}. \tag{21}$$

where *m* is the plate mass, {*S*} is a line vector with the area of each boundary element, {1} is a column vector filled with ones, and *P* is the total force applied on the plate.

5 NUMERICAL RESULTS

In this section, some results are presented in their normalized form. Thus, it is necessary to show how is the arrangement of the relative stiffness between, the plate and the soil, the dimensionless frequency factor, the mass ratio and the magnification factor. The relative stiffness is expressed through the dimensionless factor γ, relating the Young's modulus of the soil (isotropic case) and the plate, whereby:

$$\gamma = 180\pi \frac{1-\nu_p^2}{1-\nu_s^2} \frac{E_s}{E_p} \frac{L_x^2 L_y}{t^3}. \tag{22}$$

where t is the thickness of the plate, E_s and E_p are the Young's modulus of the soil and the plate, ν_s and ν_p are the Poisson's ratio of the soil and the plate and L_x and L_y are the side lengths of the plate.

The frequency is expressed through the dimensionless factor a_0, as it is shown below:

$$a_0 = \frac{L_x}{2}\omega\sqrt{\frac{\rho_s}{c_{44}}}. \tag{23}$$

where ω is the circular frequency of the dynamic load and ρ_s is the soil density.

The plate mass is given by a non-dimensional mass ratio B_z:

$$B_z = \frac{m}{\rho_s \left(\frac{L_x}{2}\right)^2 \left(\frac{L_y}{2}\right)} \frac{1-\nu_s}{4\alpha}. \tag{24}$$

where m is the total mass of the plate and α is the shape factor. For square plates $\alpha = 1{,}065$.

The magnification function V is the dynamic displacement amplitude divided by the displacements in the z direction for the static case of the rigid plate for the same load. For the analysis, some values are chosen according to the literature of what is the most relevant in machine foundations. Thus, the dimensionless frequency varies from 0 to 4 and three different relative stiffness were used: rigid plate, $\log_{10}\gamma = 0{,}36$, flexible plate, $\log_{10}\gamma = 3{,}36$, and very flexible plate, $\log_{10}\gamma = 4{,}27$. For the mass ratio, $B_z = 2.00$ is used.

A square plate is analyzed, with side length $L_x = L_y$. The plate is divided along the x and y axes into 8 elements. For each plate element, there is a soil nodal point, counting 64 plate and soil elements. A concentrated load in the center of the plate is applied and the magnification factor is measured at the middle (V_M) and at the corner (V_E) of the plate.

The results in Figure 3 compares the results obtained by Savidis and Ritcher (1979), in which another formulation is used.

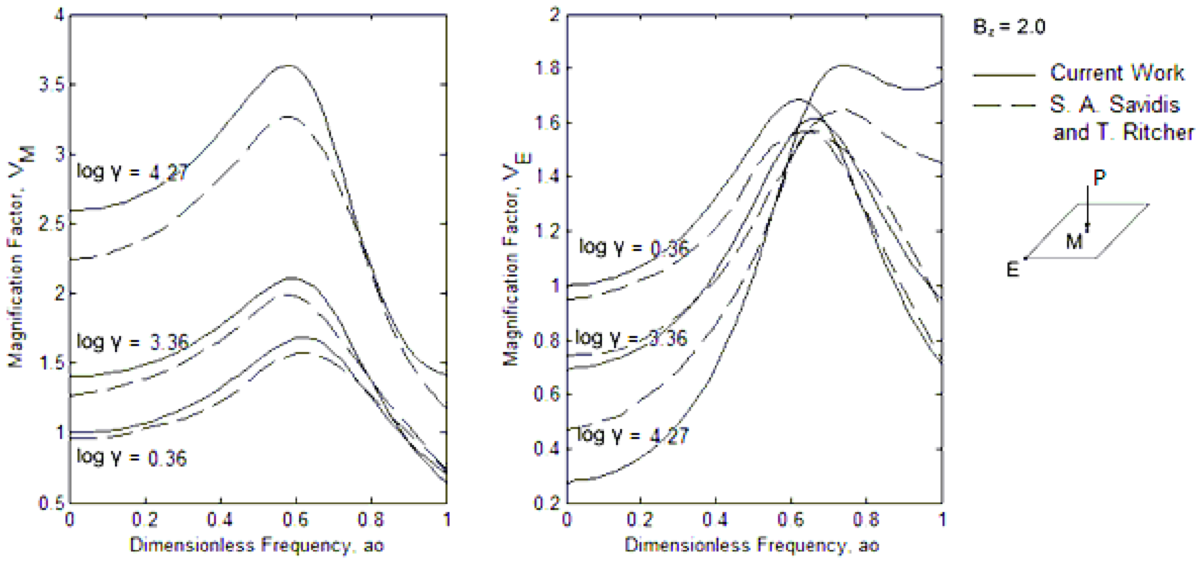

Figure 2. Comparison between two different formulations

The two formulations lead to similar behavior for a dynamic load. For rigid plate, the results are very close, but the more the plate increases the flexibility, more different the results are. After comparing the two different formulations, the anisotropic effect is analyzed. The isotropic and the anisotropic soils elastic constants are shown in table 2.

Table 2. Characteristics of the materials

Material	c_{11}/c_{44}	c_{13}/c_{44}	c_{33}/c_{44}	n_1	n_3
Isotropic ($\nu = 0.25$)	3	1	3	1	1
Anisotropic	6	2.648	3	0.5	0.378

The isotropic material has a Poison ratio ν = 0.25 and the anisotropic material is chosen to have approximately the same vertical stiffness and the same Rayleigh wave-speed of the isotropic material. The damping factor was incorporated into the influence functions evaluation by considering complex valued elastic constants.

The plate stiffness and the soil stiffness are two components that influence the displacement w. Therefore, the displacement is divided in two parts: $w = w_r + w_f$. The first part w_r is the displacement of a rigid plate, which depends on the soil properties and on the plate dimensions. This part can be expressed in a dimensionless form:

$$\bar{w}_r = \frac{w_r c_{44} L_x}{P}. \qquad (25)$$

where P is the concentrated load value.

The second part w_f is the plate flexibility effect, which depends on the plate stiffness and on the dimensions. Then, it can be represented by:

$$\bar{w}_f = \frac{w_f D}{P L_x^{\ 2}}. \tag{26}$$

where $D = E_p t^3 / [12(1-\nu_p^2)]$ is the stiffness of the plate and P is the concentrated load. The displacement w also depends on the mass that is represented by the mass ratio B_z.

In terms of the dimensionless parameters, the results comparing the isotropic and the anisotropic soil are presented below, where the displacement $\bar{w}_f$, the contact pressure $\bar{q} = q/(L_x L_y)$ and the plate bending moments $\bar{m}_x = m_x / P$ and $\bar{m}_y = m_y / P$ are evaluated for one dimensionless frequency $a_0 = 1$ and three different relative stiffness used in the previous results. In order to get more accurately the anisotropic effect, the plate of side lengths $L_x = L_y$ is divided into 20 elements in both sides, resulting 400 elements. The results are measured alongside a straight line from the center to the border of the plate.

The plots in Figure 4 show the displacement of a rigid plate with $B_z = 2$ for the two types of soils. The effect of anisotropy is relatively small, but distinctive. It is worth noting that once the displacement for a particular mass value is obtained, the displacement for any other mass can be easily obtained.

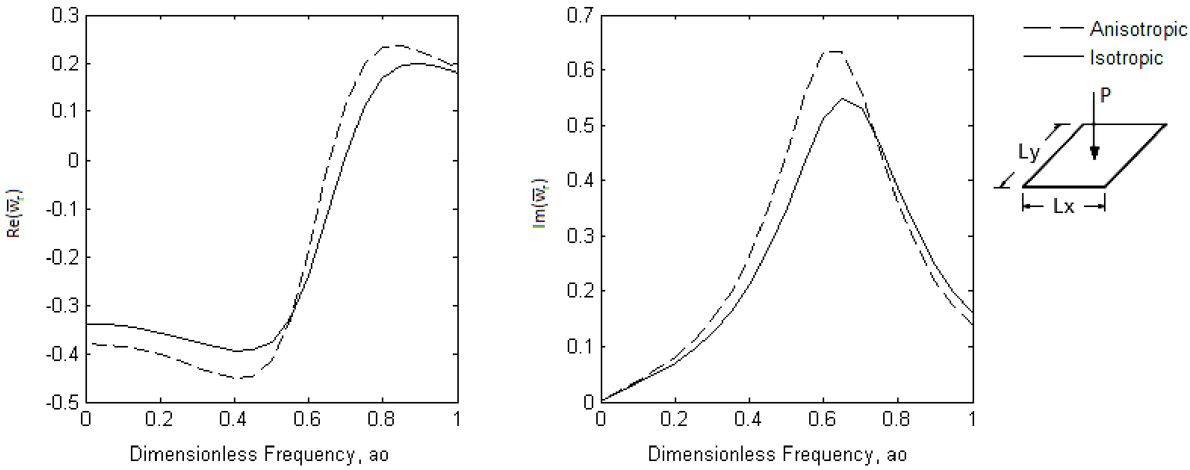

Figure 4. Comparison between the values of the real and imaginary parts measured at the center of the plate of the dimensionless rigid displacements (w_r) for different frequencies.

The plots in Figures 5-7 show the normalized displacement $\bar{w}_f$, interface stress $\bar{q}$ and moments $\bar{m}_x$ and $\bar{m}_y$ profiles along $\bar{x} = x / L_x$ for $a_0 = 1$ and three different relative stiffness γ. The effect of anisotropy is very clear in all components.

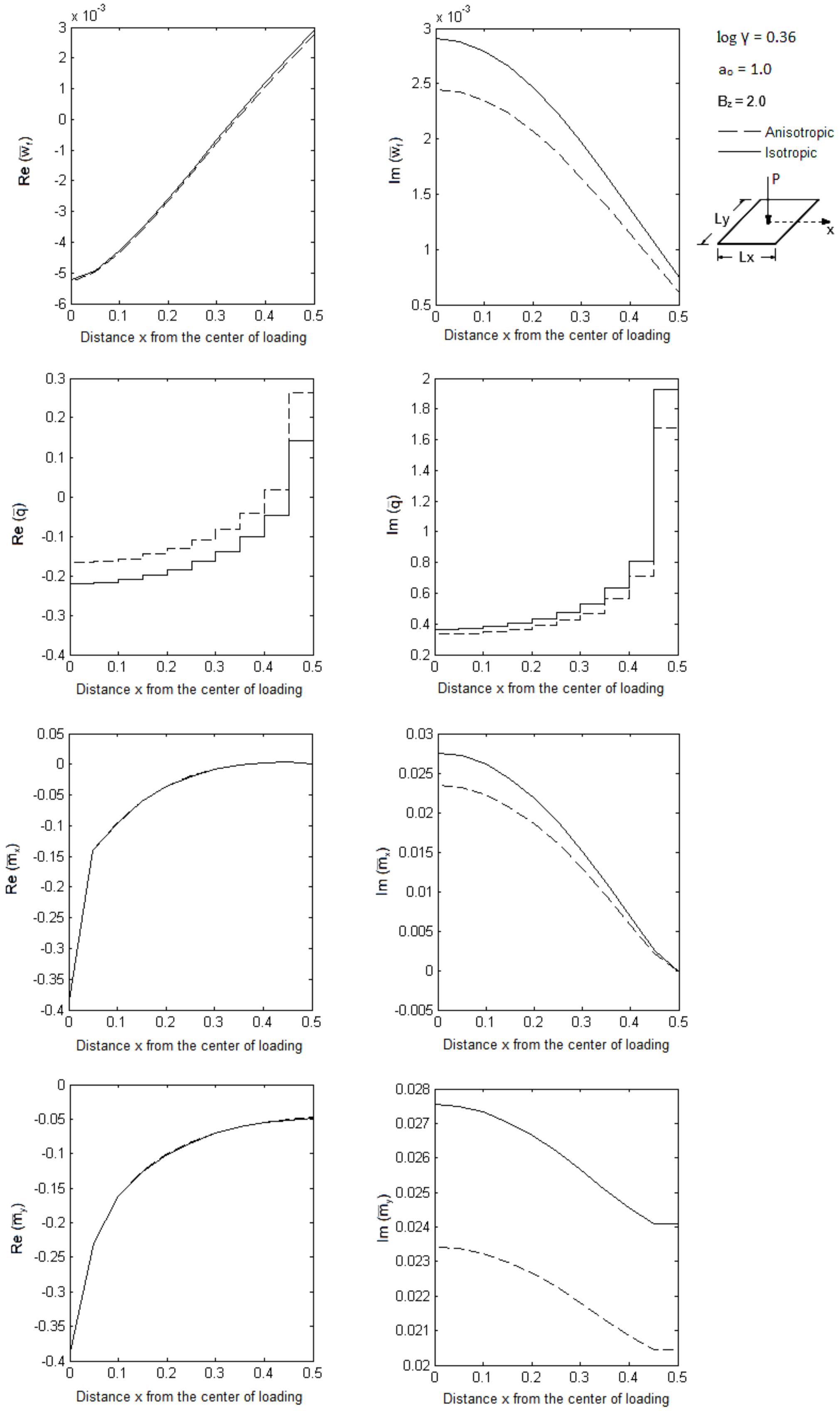

Figure 5. Comparison between the values of the real and imaginary parts of the dimensionless displacements (flexible effect), contact pressure and moments in x and y direction for a rigid plate

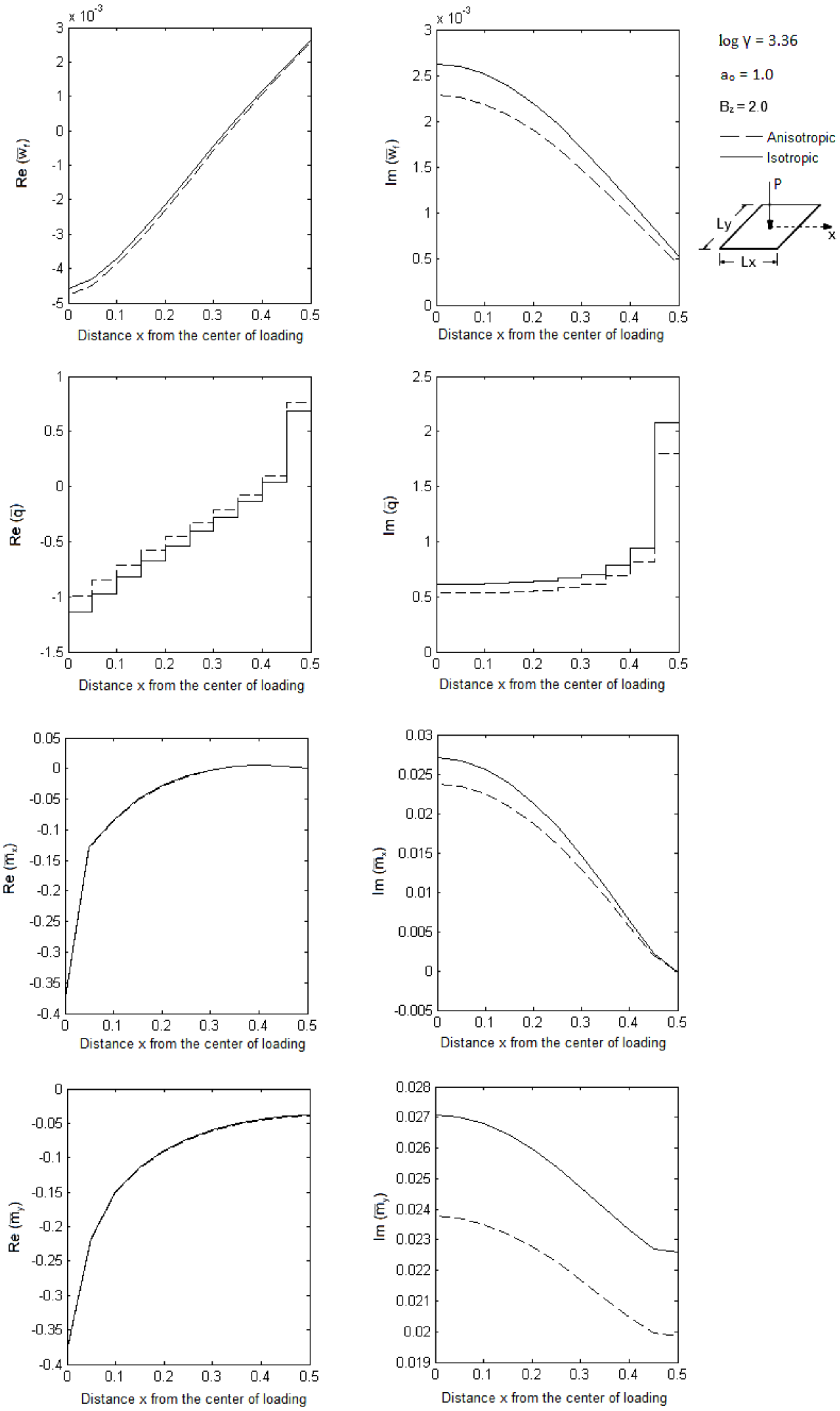

Figure 6. Comparison between the values of the real and imaginary parts of the dimensionless displacements (flexible effect), contact pressure and moments in x and y direction for a flexible plate

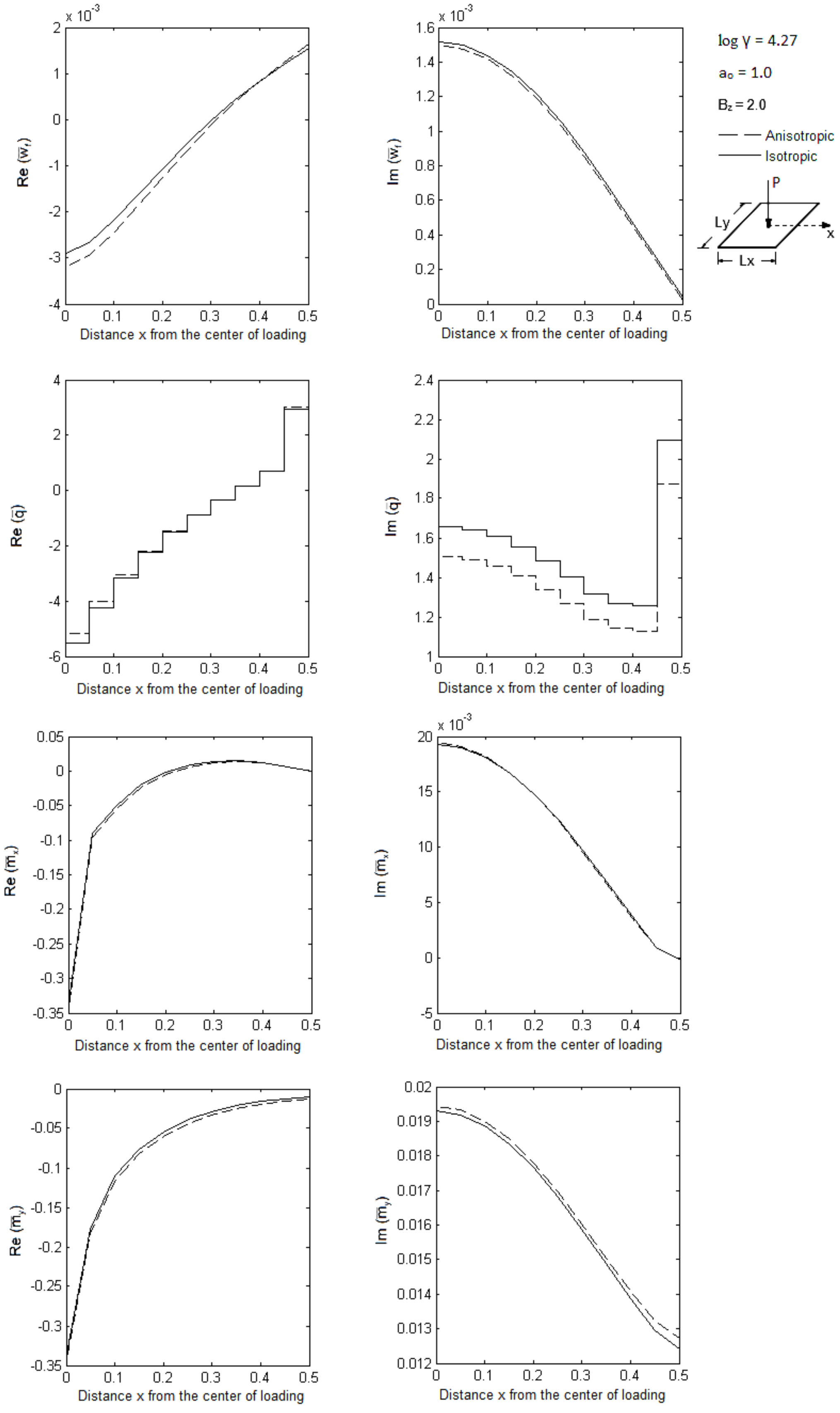

Figure 7. Comparison between the values of the real and imaginary parts of the dimensionless displacements (flexible effect), contact pressure and moments in x and y direction for a very flexible plate

6 CONCLUSIONS

A numerical method for the analysis of flexible plates supported on homogeneous soil, using an indirect formulation of the Boundary Element Method with half-space influence functions was presented. The method was applied to a set of different cases of plate stiffness and soil characteristics, comparing with results in the literature.

The comparison between the anisotropic and isotropic soil for the chosen frequency, in the analyzed cases shows that the anisotropy had more influence in the rigid plate and the soil pressure varied according to the relative stiffness.

The formulation used is relatively simple and does not require great computational effort. It may be used different shapes for the plate according to the demanded of a real problem, not necessarily rectangular.

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