REVISTA INTERDISCIPLINAR DE PESQUISA EM ENGENHARIA

CILAMCE 2016

XXXVII IBERIAN LATIN AMERICAN CONGRESS
ON COMPUTATIONAL METHODS IN ENGINEERING
BRASÍLIA - DF - BRAZIL

On the solution of 3D frictional contact problems with Boundary Element Method and discontinuous elements using a Generalized Newton Method with line search

Cristiano J.B. Ubessi

Rogério J. Marczak

cristiano.ubessi@ufrgs.br

rato@mecanica.ufrgs.br

Departamento de Engenharia Mecânica - Universidade Federal do Rio Grande do Sul - RS

Sarmento Leite 425, 900.00-000, Porto Alegre , RS, Brazil

Abstract.

This paper presents the implementation of an algorithm for the solution of 3D elastic contact problems with friction using the Boundary Element Method (BEM) with discontinuous elements. A standard BEM implementation is used, and the coupling of the potential contact zone is imposed through a projection function which treats each region independently, and is updated along with the changes to the contact state. The contact restrictions are fulfilled through the augmented Lagrangian, and the solution is found using the Generalized Newton Method with line search. With this method is possible to avoid the calculation of the non-linear derivatives, allowing for a fast solution of the problem. A classic contact problem is solved to evaluate the accuracy of the method and to provide a comparison with analytic solution.

Keywords: *Frictional Contact, Boundary Element Method, Discontinuous Elements, Generalized Newton Method with line search*

1 INTRODUCTION

Contact type problems are too often found in engineering applications. While some could be simplified or even assumed to be irrelevant, there exists cases where it is the reason of existence of the engineering problem by itself. Wear, tear, fatigue, friction, among others, are all problems which could arise by the contact occurrence. With the fast growth on the use of advanced and high performance materials in engineering, rises the need to predict the contact conditions when using these materials.

Also called the Signorini problem, since the works published by Fichera (1963), Fichera (1973), the simple unilateral contact problem imposes an ambiguous boundary condition, since the contact area and, hence the equilibrium configuration, may have a non linear dependency of the loads acting on the structure. On its series of works, the author have obtained analytical solutions for an elastic sphere lying on a rigid plane, without and with friction, considering gravitational loads.

One of the difficulties on deriving and solving frictional contact problems is that they are governed by a multivalued tribological law which does not derive from a natural potential (even non-differentiable), meaning they cannot be formulated as standard optimization problems using inequality constraints (Alart & Curnier, 1991).

The Boundary Element Method is well known for its ability to solving contact problems, since its formulation intrinsically treats the displacements and tractions with same order of approximation. This enables the direct application of the contact constraints without the need of penalty parameter or Lagrangian multipliers. Since the pioneer work of Andersson (1981), which used that property to develop an incremental loading technique to solve contact problems on 2D, a few other works are also found on the literature using the same principles, such as Andersson (1981), Paris & Garrido (1989), Garrido et al. (1991), Man et al. (1993a), Man et al. (1993b) and Paris et al. (1995). Another works are found such as Yamazaki et al. (1994), Rodríguez-Tembleque et al. (2008), which use the Lagrangian multiplier or the penalty parameter methods, which are mandatory to treat contact problems with FEM. Though they could be used, are not needed to treat contact with pure BEM discretization.

More recently, motivated by González et al. (2008), which treated FEM-BEM coupled problems, Rodríguez-Tembleque & Abascal (2010), based on the works of Pang (1990), Alart & Curnier (1991), used the Augmented Lagrangian formulation, which circumvents some weaknesses existing on Lagrangian multiplier and penalty methods, and which convergence is independent of the penalization parameter used on it. The contact restrictions are imposed in the form of projection functions, resulting on a very robust framework to both FEM and BEM frictional contact analysis. The resulting non linear system of equations is then solved by the Generalized Newton Method with line search (GNMls), which is simply a generalization of the standard Newton method to B-differentiable functions, with an unconstrained optimization between each step to accelerates its convergence. The resulting equations can be further simplified with the properties of complementarity of normal tractions and gap, reducing the number of DOFs needed on the SLE solution. The method was also used by Rodriguez-Tembleque et al. (2011) to study 3D frictional contact on anisotropic media using BEM.

This paper describes part of our research on contact analysis using BEM, where the GNMls was implemented on the resolution of contact problems using discontinuous elements. The paper is presented on the following methodology: First the contact problem is described along with the restrictions it imposes. The boundary integral formulation leading to BEM equations is presented. The contact variables on the discrete form, along with the projection functions and the Augmented Lagrangian variables which imposes the restrictions on the discrete form. The resulting non linear system of equations is presented with an analogy to problems with multiple regions . The GNMls is described with the linearized Jacobian. On Results section, the classical Hertz contact problem, considering two elastic regions, is analyzed using two BEM meshes and the solution obtained is compared with the analytical one. Final considerations are made, which closes the present work.

2 METHODOLOGY

2.1 The elastic contact problem

The problem of contact between two linear elastic bodies, is a problem which occurs at the boundary, between two bodies or two regions of same body, and a problem where the linear elasticity equations remains valid. Consider the simple problem of two separated elastic bodies, which may come in to contact, as illustrated on Fig. 1. Treating this as a classical Boundary Value Problem (BVP), one knows by anticipation the prescribed conditions at the bodies boundaries: Tractions are null on the contours which are free to displace in any direction, and are unknowns on this regions (Γ_{t0}); Displacement are generally prescribed as null in some directions, also called restrictions, but can also be different from zero, and in both cases the tractions will be unknowns on those regions. The contact boundary, Γ_c have conditions which depends on the contact state, defined by the distance between the two bodies, which cannot be negative. When it is positive, the surfaces are free and the tractions are null. Compatibility conditions must be set when the distance is zero.

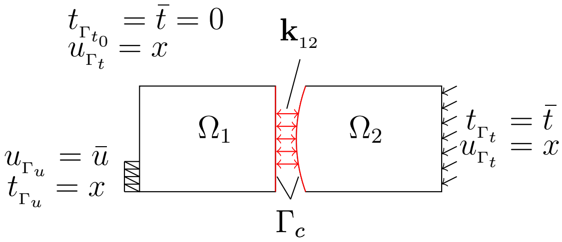


Figure 1: Solid under consideration

The conditions relate the displacement and the tractions on the contact surface, and will depend on the existence of friction or not.

Kinematic variables

In this work the BEM formulation is assuming small strains and displacements, and node on node contact. The nodes are assumed to be positioned in a conforming scheme, i.e., the slave nodes are positioned as closely as possible to the master, or matching the displacement path performed by the contact pair. The contact variables in the discrete form will then be related to the possible contact node pairs. The contact frame is based on the master nodes, and the gap variable g is obtained through the following relation

$$\mathbf{g} = \mathbf{B}^T(\mathbf{x}_2 - \mathbf{x}_1) + \mathbf{B}^T(\mathbf{u}_2 - \mathbf{u}_1), \tag{1}$$

where $\mathbf{x}_1$ and $\mathbf{x}_2$ are master and slave nodal coordinates, $\mathbf{u}_1$ and $\mathbf{u}_2$ are the respective nodal displacement vectors, on the global Cartesian coordinate system, and $\mathbf{B}$ is a change of base matrix constructed with the three unit vectors which form a local coordinate system with origin at $\mathbf{x}_1$, the master node position, i.e.,

$$\mathbf{B} = \begin{bmatrix} \mathbf{t}_1 & \mathbf{t}_2 & \mathbf{n} \end{bmatrix}. \tag{2}$$

2.2 Boundary Element Method formulation

The BEM formulation derived for a homogeneous media with tractions and displacements prescribed on its boundary, as shown in Fig. 2, starting from the elasticity equilibrium equations:

$$\begin{cases} \sigma_{kj,j} + b_k = 0 & \text{in } \Gamma \\ \sigma \cdot n = t & \text{in } \Gamma \\ u_{\Gamma_u} = \bar{u} & \text{in } \Gamma_u \\ t_{\Gamma_t} = \bar{t} & \text{in } \Gamma_t. \end{cases} \tag{3}$$

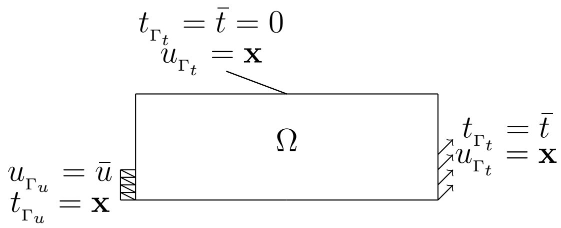


Figure 2: Solid under consideration.

By means of the principle of virtual work, is possible to obtain the Somigliana's well known identity (Brebbia et al., 2012):

$$u_l^i(\mathbf{y}) + \int\limits_{\Gamma} p_{lk}^*(\mathbf{x}, \mathbf{y}) u_k(\mathbf{x}) d\Gamma = \int\limits_{\Gamma} u_{lk}^*(\mathbf{x}, \mathbf{y}) p_k(\mathbf{x}) d\Gamma + \int\limits_{\Omega} u_{lk}^*(\mathbf{x}, \mathbf{y}) b_k(\mathbf{x}) d\Omega, \tag{4}$$

where y is the source point inside the domain and x represents the field points on the boundary. The symbols u^* and $p^*(\mathbf{x}, \mathbf{y})$ are the fundamental solutions for the displacement or traction in the direction k on point $\mathbf{x}$ when a unit load is applied in the direction l at point $\mathbf{y}$. When the source point is taken to the boundary, a limit has to be taken, which leads to a free term c_{lk}^i, which will depend on the shape of the boundary at $\mathbf{y}$, resulting in the following BIE:

$$c^i_{lk}u^i_k(\mathbf{y}) + \int\limits_{\mathbf{\Gamma}} \mathbf{p^*_{lk}}(\mathbf{x},\mathbf{y})\mathbf{u_k}\mathbf{d\Gamma} = \int\limits_{\mathbf{\Gamma}} \mathbf{u^*_{lk}}(\mathbf{x},\mathbf{y})\mathbf{p_k}\mathbf{d\Gamma} + \int\limits_{\mathbf{\Omega}} \mathbf{u^*_{lk}}(\mathbf{x},\mathbf{y})\mathbf{b_k}\mathbf{d\Omega}. \tag{5}$$

where, in the case of this work, the boundary at $\mathbf{y}$ is always smooth due to the use of discontinuous elements, i.e., $c^i_{lk} = \frac{1}{2}$.

To obtain a numerical solution from Eq. (5), the boundary is discretized in to a finite number of elements, forming an algebraic system of equations. The geometry of the body will be calculated by means of shape functions in terms of the parametric coordinates $\boldsymbol{\xi} = (\xi_1, \xi_2)$:

$$\mathbf{x} = \sum_{n=1}^{N} \phi_n(\boldsymbol{\xi})x^j_n, \tag{6}$$

where N is the number of nodes of the element and ϕ_n is a vector containing the geometric shape functions. To simplify the discretization and formulation of the problem, discontinuous elements are used, and so the physical variables of the problem will be calculated in physical nodes that are offset relative to the geometrical nodes. Displacements and tractions will be interpolated with the discontinuous interpolation functions, $\bar{\phi}$:

$$\mathbf{u} = \sum_{n=1}^{N} \bar{\phi}_n(\boldsymbol{\xi})u^j_n, \qquad \mathbf{p} = \sum_{n=1}^{N} \bar{\phi}_n(\boldsymbol{\xi})p^j_n. \tag{7}$$

In this work, discontinuous rectangular elements were used, and the interpolation functions for the linear element are

$$\bar{\phi}_n = \frac{1}{4d_o^2}(d_0 + \xi_1\bar{\xi}^n_1)(d_0 + \xi_2\bar{\xi}^n_2), \quad n = 1\ldots4, \tag{8}$$

where $(\bar{\xi}^n_1, \bar{\xi}^n_2)$ are the nodal coordinates of the n-th geometrical node (e.g., $(\bar{\xi}^1_1, \bar{\xi}^1_2) = (-1,-1)$), (ξ_1, ξ_2) are the coordinates where the function is being calculated, and d_0 is the distance from the center of the element to the nodes in the local coordinates Beer et al. (2008). This distance can be related to the percentile offset (a) by:

$$d_0 = \frac{(1-\mathrm{a})}{100}, \tag{9}$$

which returns the continuous functions ϕ_n when a $= 0$. Following the same notation as the latter, the interpolation functions for the 8-node element could be summarized on the following equations:

$$\begin{aligned} \bar{\phi}_n &= \frac{1}{4d_o^3}(d_0 + \xi_1\bar{\xi}^n_1)(d_0 + \xi_2\bar{\xi}^n_2)(\xi_1\bar{\xi}^n_1 + \xi_2\bar{\xi}^n_2 - d_o), \quad n = 1\ldots4, \\ \bar{\phi}_n &= \frac{1}{2d_o^3}\frac{(d_0 + \xi_1)(d_0 - \xi_1)(d_0 + \xi_2)(d_0 - \xi_2)}{(d_o - \xi_1\bar{\xi}^n_1 - \xi_2\bar{\xi}^n_2)}, \qquad n = 5\ldots8. \end{aligned} \tag{10}$$

Writing Eq. (5) in matrix notation, including the Jacobian J (omitting the body forces) results in the following equation:

$$\mathbf{C}^i\mathbf{u}^i + \sum_j \left\{ \int_{\Gamma_\xi} \mathbf{p}^{*ij}\boldsymbol{\Phi} J d\Gamma_\xi \right\} \mathbf{u}^j = \sum_j \left\{ \int_{\Gamma_\xi} \mathbf{u}^{*ij}\boldsymbol{\Phi} J d\Gamma_\xi \right\} \mathbf{p}^j \tag{11}$$

where the summation over j spans over all elements. With discontinuous elements, the free term coefficients matrix will be $c^i = \frac{1}{2}\mathbf{I}$ when the node belongs to the element being integrated, and null otherwise. Equation (11) is valid for a source node i, and further combination of the terms computed after a collocation process over all boundary nodes will result in the algebraic system of equations that leads to the BEM solution, i.e.,

$$\mathbf{H}\mathbf{u} = \mathbf{G}\mathbf{t}. \tag{12}$$

2.3 Contact constraints

The well known frictional contact restrictions are:

$$\begin{cases} \mathrm{t}_n \leq 0; \quad \mathrm{g}_n = 0; \quad \dot{\mathbf{g}}_t = 0; & \text{Contact - Stick,} \\ \mathrm{t}_n \leq 0; \quad \mathrm{g}_n = 0; \quad \|\mathbf{t}_t\| = \mu\,|t_n|\,; \quad \dot{\mathbf{g}}_t \cdot \dot{\mathbf{t}}_t = -\,\|\dot{\mathbf{g}}_t\|\,\left\|\dot{\mathbf{t}}_t\right\|; & \text{Contact - Slip,} \\ \mathrm{t}_n = 0; \quad \mathrm{g}_n \geq 0; \quad \mathbf{t}_t = 0; & \text{Separated.} \end{cases} \tag{13}$$

The frictional contact law is fulfilled by means of projection operators, i.e. functions which project the the contact variables in to the admissible solution region (Rodríguez-Tembleque & Abascal, 2010).

Normal operator

The normal tractions projector function takes the form: $\mathbb{P}_{\mathbb{R}_-}(\cdot) : \mathbb{R} \rightarrow \mathbb{R}_-$, $\mathbb{P}_{\mathbb{R}_-}(x) = \min(x, 0)$. The mixed variable, *augmented normal traction* t_n^* is defined as: $t_n^* = t_n + r_n \mathrm{g}_n$, where $r_n \in \mathbb{R}_+$ is a positive penalization parameter.

Tangential operator

The tangential projector function takes the form: $\mathbb{P}_{\mathbb{C}_g}(\cdot) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$,

$$\mathbb{P}_{\mathbb{C}_g}(\mathbf{x}) = \begin{cases} \mathbf{x} & \text{if } \|\mathbf{x}\| < |\mu t_n|\,, \\ |\mu t_n|\,\mathbf{e}_t & \text{if } \|\mathbf{x}\| \geqslant |\mu t_n|\,, \end{cases} \tag{14}$$

where $\mathbf{e}_t = \mathbf{x}/\,\|\mathbf{x}\|$. The tangential contact restriction then is written as

$$\mathbf{t}_t - \mathbb{P}_{\mathbb{C}_g}(\mathbf{t}_t^*) = 0, \tag{15}$$

and the augmented tangential traction is defined as: $\mathbf{t}_t^* = \mathbf{t}_t - r_t\dot{\mathbf{g}}_t$ where the positive penalization parameter $r_t \in \mathbb{R}_+$ could differ from the normal one.

Normal-Tangential operator

The constraints of the combined normal-tangential contact problem can be formulated as: $\mathbf{t} - \mathbb{P}_{C_f}(\mathbf{t}^*) = 0$. The contact operator $\mathbb{P}_{C_f}$ is then defined as

$$\mathbb{P}_{C_f}(\mathbf{t}^*) = \begin{bmatrix} \mathbb{P}_{C_\mathrm{g}}(\mathbf{t}_t^*) \\ \mathbb{P}_{\mathbb{R}_-}(\mathbf{t}_n^*) \end{bmatrix}, \tag{16}$$

where the region $\mathbb{C}_g$ is the augmented friction circle with radius $\left|\mu \mathbb{P}_{\mathbb{R}_-}(\mathbf{t}^*{}_n)\right|$.

2.4 Contact treatment with Boundary Element Discretization

One of the well known advantages of BEM in contact problems is that the contact tractions are already part of the unknowns. The SLE for the contact problem will be formed by in a similar mode as it is done for multiple-region problems (e.g., Brebbia et al. (2012)), i.e., considering the geometry illustrated on Fig. 1, the following system arise,

$$\begin{aligned} \mathbf{H}^{\Gamma 1}\mathbf{u}^{\Gamma 1} - \mathbf{G}^{\Gamma 1}\mathbf{t}^{\Gamma 1} + \mathbf{H}^{\Gamma 1c}\mathbf{u}^{\Gamma 1c} + \mathbf{G}^{\Gamma 1c}\mathbf{t}^{\Gamma 1c} = \bar{\mathbf{G}}^{\Gamma 1}\bar{\mathbf{t}}^{\Gamma 1} - \bar{\mathbf{H}}^{\Gamma 1}\bar{\mathbf{u}}^{\Gamma 1} \\ \mathbf{H}^{\Gamma 2}\mathbf{u}^{\Gamma 2} - \mathbf{G}^{\Gamma 2}\mathbf{t}^{\Gamma 2} + \mathbf{H}^{\Gamma 2c}\mathbf{u}^{\Gamma 1c} - \mathbf{G}^{\Gamma 2c}\mathbf{t}^{\Gamma 1c} = \bar{\mathbf{G}}^{\Gamma 2}\bar{\mathbf{t}}^{\Gamma 2} - \bar{\mathbf{H}}^{\Gamma 2}\bar{\mathbf{u}}^{\Gamma 2} \end{aligned} \tag{17}$$

where the bonded connections on the interface between the two regions was set by the displacement and traction compatibility conditions.

Equation (17) is sufficient to calculate a bonded problem, where is assumed the interface region support tractions in all directions and remains constant. To incorporate contact restrictions on a similar equation system one have to write Eq. (17), along with two additional sets of equations: The kinematic relations which arise from the gap, i.e., Eq. (1), and the projection operators, which represent the contact restrictions, depending on the contact state of the node pair, resulting in a system $\mathbf{\Theta}(\mathbf{z}) = \mathbf{R}\mathbf{z} - \mathbf{f}$, composed by the following equation

$$\begin{bmatrix} \mathbf{A}^{\Gamma 1} & \mathbf{0} & \mathbf{A}_p^1\tilde{\mathbf{C}}^1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^{\Gamma 1} & -\mathbf{A}_p^2\tilde{\mathbf{C}}^2 & \mathbf{0} \\ \mathbf{C}^{1^\mathrm{T}} & -\mathbf{C}^{2\,T} & \mathbf{0} & \mathbf{C}_\mathrm{g} \\ \mathbf{0} & \mathbf{0} & \mathbf{P}_\lambda & \mathbf{P}_\mathrm{g} \end{bmatrix} \begin{Bmatrix} \mathbf{x}^1 \\ \mathbf{x}^2 \\ \mathbf{\Lambda} \\ \mathbf{k} \end{Bmatrix} = \begin{Bmatrix} \mathbf{b}^1 \\ \mathbf{b}^2 \\ \mathbf{C}_\mathrm{g}\mathbf{k}_\mathrm{go} \\ \mathbf{0} \end{Bmatrix}, \tag{18}$$

where $\mathbf{A}^{\Gamma n}$ are the matrices relative to the independent unknowns (mixed tractions and displacements) for the nth region, including the displacement unknowns on the possible contact region, $\mathbf{A}_p^n$ are the matrices relative to the traction unknowns on the possible contact region. The tractions on the contact interface ($\mathbf{\Lambda}$) as well as the gap ($\mathbf{k}$) are considered on their local coordinate system, by the incorporation of the rotation matrices on the system of equations, i.e., Eq. (2), which are assembled for each contact pair on the main rotation matrix presented as $\tilde{\mathbf{C}}^n$ on Eq. (18). The $\mathbf{C}^n$ matrices are the same rotation matrices, but assembled only on the positions of $\mathbf{A}^{\Gamma n}$ corresponding to displacement unknowns of the contact regions. the matrix $\mathbf{C}_g = \mathbf{I}$, and $\mathbf{k}_{\text{go}}$ is the initial gap. The projector matrices are also assembled for each contact pair, and will depend on the contact pair state:

- Free: $\lambda^*_{n\,I} \geqslant 0$

$$(\mathbf{P}_\lambda)_I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_I, (\mathbf{P}_{\text{g}})_I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_I, \tag{19}$$

- Stick: $\lambda^*_{n\,I} < 0$ and $\|\lambda^*_{t\,I}\| < \mu\,|\lambda^*_{n\,I}|$

$$(\mathbf{P}_\lambda)_I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_I, (\mathbf{P}_{\text{g}})_I = \begin{bmatrix} r_t & 0 & 0 \\ 0 & r_t & 0 \\ 0 & 0 & r_n \end{bmatrix}_I, \tag{20}$$

- Slip: $\lambda^*_{n\,I} < 0$ and $\|\lambda^*_{t\,I}\| < \mu\,|\lambda^*_{n\,I}|$

$$(\mathbf{P}_\lambda)_I = \begin{bmatrix} 1 & 0 & \mu\omega^*_{t_1} \\ 0 & 1 & \mu\omega^*_{t_2} \\ 0 & 0 & 0 \end{bmatrix}_I, (\mathbf{P}_{\text{g}})_I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -r_n \end{bmatrix}_I, \tag{21}$$

  where $\boldsymbol{\omega}^*_{t\,I} = \boldsymbol{\lambda}^*_{t\,I}/\,\|\boldsymbol{\lambda}^*_{t\,I}\|$.

Generalized Newton Method

As suggested in Rodríguez-Tembleque & Abascal (2010), the non linear system of equations is solved using the Generalized Newton Method with line search. This method was first proposed by Pang (1990), where the mathematical properties of $\mathcal{B}$-differentiable functions are described, they essentially are functions which are non Fréchet-differentiable (F-differentiable), the major difference on those functions is the absence of linearity on the B-derivative. The author presents an example of complementarity function $H : \mathbb{R}^n \to \mathbb{R}^n$, $H(x) = min(h(x), f(x))$, which is non F-differentiable, the same case as the normal projection operator. Contrary to the Uzawa method, the GNMls does not need damping or stabilization parameters, and its convergence is independent of the penalization factor used in the augmented Lagrangian (Alart & Curnier, 1991). The GNMls algorithm could be resumed on the following steps:

1. Let $\mathbf{z}^{(0)}$ be an arbitrary initial vector, and

$$\boldsymbol{\Theta}(\mathbf{z}^{(\mathbf{n})}) = \mathbf{R}^{(n)}\mathbf{z}^{(n)} - \mathbf{f} \tag{22}$$

2. Given $\mathbf{z}^{(n)}$ with $\boldsymbol{\Theta}(\mathbf{z}^{(n)}) \neq 0$, the direction $\Delta\mathbf{z}^{(n)}$ is obtained solving:

$$\boldsymbol{\Theta}(\mathbf{z}^{(n)}) + \mathcal{B}\boldsymbol{\Theta}(\mathbf{z}^{(n)}, \Delta\mathbf{z}^{(n)}) = 0 \tag{23}$$

3. Find the first integer $m = 1, 2, \ldots$ which fulfills the decreasing error condition:

$$\boldsymbol{\Psi}(\mathbf{z}^{(n)} + \alpha\Delta\mathbf{z}^{(n)}) \leqslant (1 - 2\sigma\alpha^{(n)})\boldsymbol{\Psi}(\mathbf{z}^{(n)}),$$

with $\alpha = \beta^m$, $\beta \in (0, 1)$, $\sigma \in (0, 1/2)$, and

$$\boldsymbol{\Psi}\left(\boldsymbol{\Theta}(\mathbf{z}^{(\mathbf{n})})\right) = \frac{1}{2}\left\|\boldsymbol{\Theta}(\mathbf{z}^{(\mathbf{n})})\right\|^2. \tag{24}$$

4. New solution vector: $\mathbf{z}^{(n+1)} = \mathbf{z}^{(n)} + \alpha^n\Delta\mathbf{z}^{(n)}$
5. Finish if $\Psi(\mathbf{z}^{(n+1)}) \leq \varepsilon_1$ Otherwise n=n+1, and return to (2).

3 RESULTS

The problem analyzed in this work is the Hertz classical one, with the particularity of both solids being elastic, not so commonly seen on the literature, since generally one of the bodies is considered rigid and a half space on the plane region with BEM, in favor of a simpler solution and a closer geometry to that assumed on the closed form solution. The half space representation is numerically simpler to solve, and the use of a rigid indenter is closer to a pure prescribed displacements problem, than to an elastic contact problem, as the final displacements of the contact regions are known a priori. This classical problem also permits the evaluation of the algorithm when a small region of the solids are transferring the load, concentrating the contact stress in a small area.

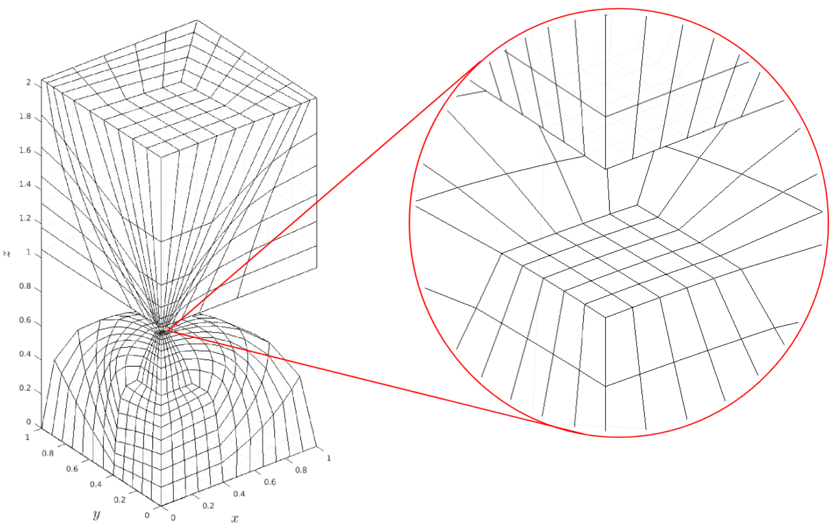


Figure 3: Mesh used on the Hertzian contact example

The mesh used for the the sphere and the cube on this problem has 888 and 468 8-node quadratic elements respectively, and is illustrated on Fig. 3. The prescribed displacements u_z

applied on the lower face of the half sphere and the cube upper face was restricted on the z direction. On all elements at the zx and zy plane a symmetry boundary condition was applied, restricting the displacements on the outward normal direction, and freeing it on the tangential direction. The offset used on the discontinuous elements (Eq. (9)) was $a = 15\%$, and all material and geometrical properties considered on this example are shown on Table 1. Also is brought the predicted maximum contact pressure from Johnson & Johnson (1987), which do not consider frictional effects on its formulation, so the coefficient of friction was set to $\mu = 0$.

Table 1: Hertz contact problem data

E	1.0×10^4	Pa	Bulk Modulus
ν	0.3	m/m	Poisson Ratio
k_0	5×10^{-2}	m	Initial separation
$r1$	1.0	m	Sphere radius
L	1.0	m	Cube length
p_0	146.3281	Pa	Maximum contact pressure
$\bar{u}_{0z}$	1.8×10^{-3}	m	Displacement applied on the sphere center

The load was applied on a single step, and the algorithm convergence is brought on Fig. 4. The initial parameters used on the GNMIs were $\beta = 0.95$, $r = 0.80$. To accelerate the solution time, an iterative sover (GMRES) was used, and a lower initial tolerance was set, along with an adaptive tolerance reduction lower than the last solution residual. As can be seen on the graph, although the solver could not converge to the desired tolerance on the final iterations, the final GNMIs residual Ψ was lower than the desired value.

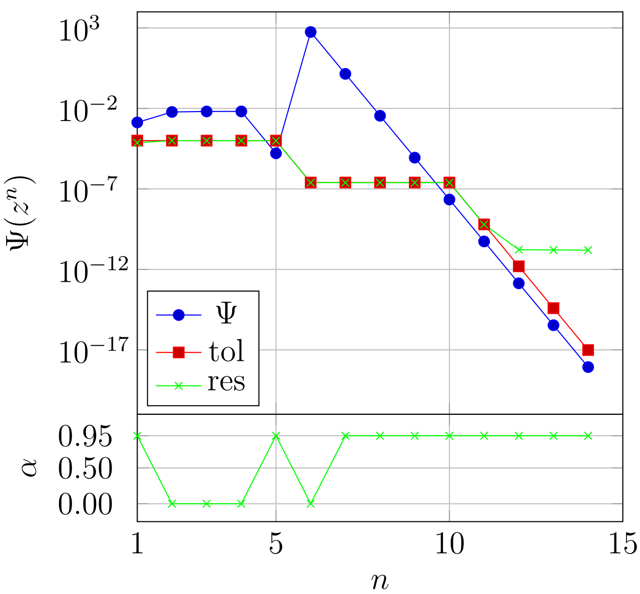
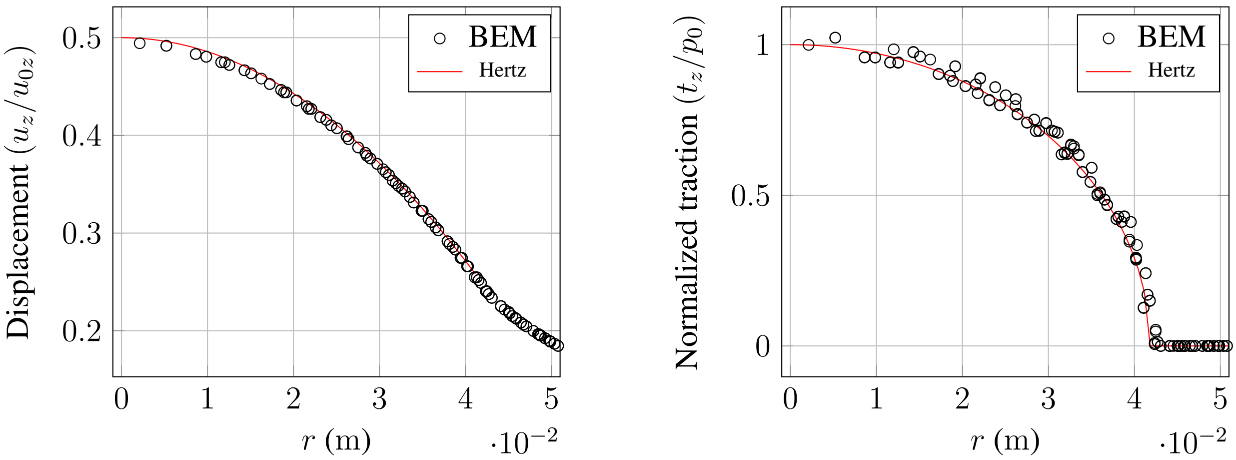


Figure 4: Newton method convergence: Residual for each iteration(Ψ), solver tolerance (tol), relative solver residual (res), and α obtained on line search.

On figure 5a, the obtained displacements on the contact region are plotted as a function of the radius from the center of the sphere along with the analytical solution. Both numerical and analytical displacements were normalized with the prescribed displacements $\bar{u}_{0z}$. The mesh nodes were disposed on a rectangular grid, so the results are scattered along their relative radius $r = \sqrt{x_i^2 + y_i^2}$. On figure 5b, the obtained normal tractions on the contact region are plotted in the same way as the displacements, and normalized with the maximum analytic contact pressure p_0. As can be seen, the normal displacements and tractions agree with the analytical solution.



(a) Displacements along contact region relative to displacement applied.

(b) Normal traction and analytical solution normalized.

Figure 5: Traction and displacements along the possible contact region selected for the problem

4 CONCLUSION

On this work it was possible to evaluate the use of discontinuous BEM to resolve a classical contact problem. A brief review of BEM contact literature was presented. The GNMls presented a good convergence rate, needing a few iterations to start converging at a logarithmic rate. Results obtained for displacements and tractions on the contact regions show a good agreement with the analytical solution.

ACKNOWLEDGEMENTS

The first author wish to express his thanks to CNPq for the doctoral scholarship.

REFERENCES

Alart, P. & Curnier, A., 1991. A mixed formulation for frictional contact problems prone to newton like solution methods. *Comput. Methods Appl. Mech. Eng.*, vol. 92, no. 3, pp. 353–375.

Andersson, T., 1981. The boundary element method applied to two-dimensional contact problems with friction. In *Boundary element methods*, pp. 239–258, Springer.

Beer, G., Smith, I., & Duenser, C., 2008. *The Boundary Element Method with Programming*. Springer Wien, New York.

Brebbia, C., Telles, J., & Wrobel, L., 2012. *Boundary Element Techniques: Theory and Applications in Engineering*. Springer Berlin Heidelberg.

Fichera, G., 1963. Sul problema elastostatico di Signorini con ambigue condizioni al contorno. *Atti Accad. Naz. Lincei, VIII. Ser., Rend., Cl. Sci. Fis. Mat. Nat.*, vol. 34, pp. 138–142.

Fichera, G., 1973. Boundary value problems of elasticity with unilateral constraints. In *Linear Theories of Elasticity and Thermoelasticity*, pp. 391–424, Springer.

Garrido, J., Foces, A., & Paris, F., 1991. B.e.m. applied to receding contact problems with friction. *Mathematical and Computer Modelling*, vol. 15, no. 3, pp. 143 – 153.

González, J. A., Park, K., Felippa, C. A., & Abascal, R., 2008. A formulation based on localized lagrange multipliers for bem–fem coupling in contact problems. *Computer Methods in Applied Mechanics and Engineering*, vol. 197, no. 6, pp. 623–640.

Johnson, K. L. & Johnson, K. L., 1987. *Contact mechanics*. Cambridge university press.

Man, K., Aliabadi, M., & Rooke, D., 1993a. Bem frictional contact analysis: load incremental technique. *Computers & structures*, vol. 47, no. 6, pp. 893–905.

Man, K., Aliabadi, M., & Rooke, D., 1993b. Bem frictional contact analysis: modelling considerations. *Engineering analysis with boundary elements*, vol. 11, no. 1, pp. 77–85.

Pang, J.-S., 1990. Newton's method for b-differentiable equations. *Mathematics of Operations Research*, vol. 15, no. 2, pp. 311–341.

Paris, F., Blazquez, A., & Canas, J., 1995. Contact problems with nonconforming discretizations using boundary element method. *Computers & structures*, vol. 57, no. 5, pp. 829–839.

Paris, F. & Garrido, J., 1989. An incremental procedure for friction contact problems with the boundary element method. *Engineering Analysis with Boundary Elements*, vol. 6, no. 4, pp. 202 – 213.

Rodríguez-Tembleque, L. & Abascal, R., 2010. A fem-bem fast methodology for 3d frictional contact problems. *Comput. Struct.*, vol. 88, no. 15-16, pp. 924–937.

Rodriguez-Tembleque, L., Buroni, F., Abascal, R., & Sáez, A., 2011. 3d frictional contact of anisotropic solids using bem. *European Journal of Mechanics-A/Solids*, vol. 30, no. 2, pp. 95–104.

Rodríguez-Tembleque, L., González, J. Á., & Abascal, R., 2008. A formulation based on the localized lagrange multipliers for solving 3d frictional contact problems using the BEM . *Numerical Modeling of Coupled Phenomena in Science and Engineering: Practical Use and Examples*, p. 359.

Yamazaki, K., Sakamoto, J., & Takumi, S., 1994. Penalty method for three-dimensional elastic contact problems by boundary element method. *Computers & structures*, vol. 52, no. 5, pp. 895–903.