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## **SOME ISSUES IN THE GENERALIZED NONLINEAR EIGENVALUE ANALYSIS OF TIME-DEPENDENT PROBLEMS IN THE SIMPLIFIED BOUNDARY ELEMENT METHOD**

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***Abstract.** The third author and collaborators have combined and extended Pian's hybrid finite element formulation and Przemieniecki's suggestion of displacement-based, frequency-dependent elements to arrive at a hybrid boundary element method for the general modal analysis of transient problems. Starting from a frequency-domain formulation, it has been shown that there is an underlying symmetric, nonlinear eigenvalue problem related to the lambda-matrices of a free-vibration analysis, with an effective stiffness matrix expressed as the frequency power series of generalized stiffness and mass matrices.*

*Although the formulation is undeniably advantageous in the analysis of framed structures, for which all coefficient matrices can be analytically obtained, its practical application as a*

general finite/boundary element analysis tool is questionable. In fact, dealing with large-scale problems calls for simplifications to speed up the numerical evaluations, which unavoidably occur at the cost of the symmetry – or just positive-definiteness – of the involved matrices. These issues deserve a closer theoretical investigation both in terms of applicability of the method and of the further generalization of the underlying eigenvalue problem, whose efficient solution seems to demand the use of advanced eigenvalue-deflation techniques, among other manipulation possibilities. This is the subject of the present paper, which also includes some illustrative numerical examples.

**Keywords:** Boundary elements, Time-dependent problems, Generalized modal analysis, quasi-symmetric problems, Deflation method

## 1 INTRODUCTION

A formulation for the free vibration analysis of truss and beam elements was proposed by Przemieniecki (1968) on the basis of a frequency power series. Starting from this formulation, Dumont (2007) introduced a generalized modal superposition technique to arrive at very accurate, uncoupled differential equations for the analysis of time-dependent problems.

The underlying nonlinear eigenvalue problem is solved by generalizing the Jacobi-Davidson algorithm and thus requiring that the mass coefficient matrices be positive definite. Since owing to discretization errors this linear-algebra property cannot be attained in a general 2D or 3D formulation, it may happen that some eigenpairs become lost in the analysis, which is detrimental to the intended accuracy increase. A first attempt to solve the problem was the use of advanced eigenvalue deflation methods, which have also come out as not advantageous, as they may also incur in increasing round-off errors besides the fact that such methods seem to be hardly applicable to a general nonlinear eigenvalue analysis.

This paper deals with the linear algebra issues just outlined above as restricted to problems without damping – although viscous damping can be seamlessly considered (Dumont, 2007). Some numerical examples are displayed. A work is in progress to increase the accuracy of the numerical evaluation of the nearly singular integrals involved in the boundary element formulation (Dumont 2007) in such a way that most of the related linear algebra problems can be at least minimized.

## 2 A BRIEF OUTLINE OF THE SIMPLIFIED HYBRID BOUNDARY ELEMENT METHOD APPLIED TO TIME-DEPENDEN PROBLEMS

The equilibrium equations for stresses  $\sigma_{ij}$  in an elastodynamic problem are

$$\sigma_{ji,j} + b_i - \rho \ddot{u}_i = 0 \quad \text{in the domain } \Omega, \quad (1)$$

for body forces  $b_i$  and specific mass density  $\rho$  (damping not considered, for simplicity) and

$$\sigma_{ij} \eta_j = \bar{t}_i \quad \text{on the boundary } \Gamma_\sigma, \quad (2)$$

for prescribed traction forces  $\bar{t}_i$  on  $\Gamma_\sigma$  and such that  $\Gamma_\sigma \cap \Gamma_u = \emptyset$  and  $\Gamma_\sigma \cup \Gamma_u = \Gamma$ .

The corresponding displacements  $u_i$  must comply with prescribed boundary conditions

$$u_i = \bar{u}_i \quad \text{along } \Gamma_u. \quad (3)$$

An approximate numerical solution is assumed on the boundary as

$$\tilde{u}_i = u_{im} d_m(t) \text{ along } \Gamma \text{ such that } \tilde{u}_i = \bar{u}_i \text{ on } \Gamma_u, \quad (4)$$

where  $u_{im}$  are interpolating functions and  $d_m(t)$  are the problem's primary unknowns.

On the other hand, *field* displacements  $u_i^f$  are approximated as

$$u_i^f = u_{im}^* p_m^* + u_i^p \text{ in } \Omega, \quad (5)$$

where  $u_i^*$  – and corresponding stresses  $\sigma_{ij}^*$  – are the homogeneous solution of Eq. (1):

$$\begin{aligned} u_i^* &= u_{im}^*(t) p_m^*(t) \\ \sigma_{ij}^* &= \sigma_{ijm}^*(t) p_m^*(t) \end{aligned} \quad (6)$$

to be obtained in terms of time-dependent point force parameters  $p_m^*(t)$ , and  $u_i^p$  is a particular solution.

The simplified hybrid boundary element method starts with the above assumptions to arrive by means of a variational statement at the matrix equilibrium equation

$$\mathbf{H}^T \mathbf{p}^* = \mathbf{p} - \mathbf{p}^p \text{ with } \mathbf{H} \equiv H_{mn} = \int_{\Gamma} \sigma_{ijm}^* \eta_j u_m d\Gamma, \quad (7)$$

where  $\eta_j$  is the outward unit vector to the boundary and  $\mathbf{H}^T$  is an equilibrium matrix that transforms point force parameters  $\mathbf{p}^* \equiv p_m^*(t)$  into equivalent nodal forces

$$\mathbf{p} \equiv p_n = \int_{\Gamma} t_i u_{in} d\Gamma, \quad (8)$$

$$\mathbf{p}^p \equiv p_n^p = \int_{\Gamma} \sigma_{ij}^p \eta_j u_{in} d\Gamma. \quad (9)$$

$\mathbf{H}$  turns out to be the same double-layer potential matrix of the conventional, collocation boundary element method. For a singular fundamental solution the integral in Eq. (7) referring to a static problem (with corresponding matrix  $\mathbf{H}_0$ ) is singular and needs to be dealt with in terms of a finite part integral plus a discontinuous term.

A second equation that is inherent to the simplified hybrid boundary element method evaluates boundary nodes displacements directly in terms of the fundamental solution by using Eq. (5):

$$\mathbf{U}^* \mathbf{p}^* = \mathbf{d} - \mathbf{d}^p, \quad (10)$$

where  $\mathbf{U}^* \equiv U_{mn}^* = U_{mm}^*$  is the matrix expression of the displacement fundamental solution  $u_{im}^*(t) \equiv u_{im}^*$  of Eqs. (5) and (6) measured at boundary nodal degrees of freedom  $n$  for point forces applied at  $m$  and  $\mathbf{d}^p$  is the vector of corresponding particular displacements  $u_i^p$ . The static part  $\mathbf{U}_0^*$  of  $\mathbf{U}^*$  for  $m$  and  $n$  corresponding to the same nodal point and referring to a static problem cannot be evaluated directly (they are not infinite!) and must be obtained by forcing that these equations be valid for the solution of a given simple static problem such as for an applied linear displacement field (Dumont and Aguilar, 2012; Aguilar, 2013).

Solving for  $\mathbf{p}^*$  in Eqs. (7) and (10) one obtains

$$\mathbf{p} - \mathbf{p}^p = \mathbf{H}^T (\mathbf{U}^*)^{-1} (\mathbf{d} - \mathbf{d}^p), \quad (11)$$

where

$$\mathbf{K} = \mathbf{H}^T (\mathbf{U}^*)^{-1} \quad (12)$$

plays the role of a stiffness matrix. Differently from the matrix obtained in the (fully variational) hybrid boundary element method, the matrix of the simplified formulation, as given above, is not symmetric, although it tends to be symmetric with increasing mesh refinement and adequate numerical evaluation of the integral used in the definition of  $\mathbf{H}$  in Eq. (7). Then, in order to make use of the proposed algorithm to solve the ensuing nonlinear eigenvalue problem (Dumont, 2007), a symmetric approximation can be obtained as

$$\mathbf{K} = \frac{1}{2} \left( \mathbf{H}^T (\mathbf{U}^*)^{-1} + (\mathbf{U}^*)^{-T} \mathbf{H} \right), \quad (13)$$

although it cannot be ensured that the generalized mass matrices of the frequency domain developments of the next Section be positive definite.

### 3 FREQUENCY DOMAIN FORMULATION

A frequency domain formulation of the problem stated as in Eq. (11) with the symmetric stiffness matrix of Eq. (13) is given by writing

$$\mathbf{U}^* = \sum_{i=0}^n \omega^{2i} \mathbf{U}_i^*, \quad \mathbf{H} = \sum_{i=0}^n \omega^{2i} \mathbf{H}_i, \quad \mathbf{K} = \sum_{i=0}^n \omega^{2i} \mathbf{K}_i \quad (14)$$

as obtained for the fundamental solution of Eq. (6) expressed in the frequency domain.  $\mathbf{U}^*$  and  $\mathbf{H}$  turn out to be the above indicated power series and  $\mathbf{K}$  is evaluated by operating with power series, as given in Eq. (13).

Then, for a problem without damping,  $\mathbf{K}_0$  becomes the static stiffness matrix,  $\mathbf{K}_1 \equiv \mathbf{M}_1$  is the same mass matrix of a conventional dynamic structural analysis problem, and the matrices  $\mathbf{K}_i \equiv \mathbf{M}_i$ ,  $i > 1$ , are generalized mass matrices that should ideally be positive definite, as shown by Dumont (2007), which also deals with the more general case of viscous damping.

### 4 ADVANCED MODE SUPERPOSITION ANALYSIS

The nonlinear eigenvalue problem for a mode superposition solution of the time-dependent problem proposed in Eq. (11), for which the generalized stiffness matrix has been symmetrized according to Eq. (13), is compactly expressed as

$$\mathbf{K}_0 \Phi - \sum_{i=1}^n \mathbf{M}_i \Phi \Omega^{2i} = 0, \quad (15)$$

where  $\Omega^2$  is a diagonal matrix of eigenvalues, which is, in principle, of the same order of  $\mathbf{K}$ , and  $\Phi$  is a matrix whose columns are the corresponding eigenvectors. In an actual problem, not all eigenvalues and eigenvectors (eigenpairs) need be evaluated.

As shown by Dumont (2007), the eigenvector matrix  $\Phi$  fulfils the following orthogonality and normality properties for the proposed nonlinear problem:

$$\sum_{j=1}^n \sum_{k=1}^j \Omega^{2(k-1)} \Phi^T \mathbf{M}_j \Phi \Omega^{2(j-k)} = \mathbf{I} \quad (16)$$

$$\Phi^T \mathbf{K}_0 \Phi + \sum_{j=2}^n \sum_{k=1}^{j-1} \Omega^{2k} \Phi^T \mathbf{M}_j \Phi \Omega^{2(j-k)} = \Omega^2. \quad (17)$$

Once the nonlinear eigenvalue has been solved, the nodal displacements can be obtained in terms of the mode superposition

$$\mathbf{d} = \Phi \boldsymbol{\eta}, \quad (18)$$

where  $\boldsymbol{\eta} \equiv \boldsymbol{\eta}(t)$  is a vector of time-dependent amplitudes to be evaluated in the frame of as many uncoupled, second-order differential equations

$$\Omega^2 (\boldsymbol{\eta} - \boldsymbol{\eta}^p) + \ddot{\boldsymbol{\eta}} - \ddot{\boldsymbol{\eta}}^p = \Phi^T (\mathbf{p} - \mathbf{p}^p) \quad (19)$$

as required to adequately represent the proposed problem. The above equation is exactly the same Duhamel equation of the classical dynamic structural analysis, except that a better accuracy could in principle be achieved by means of the more refined eigenvalue solution.

#### 4.1 A linear eigenvalue arrangement with augmented matrices

The Jacobi-Davidson method (Sleijpen et al. 1996) was developed as an extension of the method firstly proposed by Davidson (1975) and later on implemented by Dumont (2007) for the solution of nonlinear eigenvalue problems. The advanced modal analysis proposed by Dumont (2007), as formulated in Eqs. (15)-(19), is mathematically equivalent to working with the linear, augmented eigenvalue arrangement

$$\left( \begin{bmatrix} \mathbf{K}_0 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 & \mathbf{M}_3 & \dots & \mathbf{M}_n \\ \mathbf{0} & \mathbf{M}_3 & \mathbf{M}_4 & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{M}_n & \mathbf{0} & \dots & \mathbf{0} \end{bmatrix} - \lambda \begin{bmatrix} \mathbf{M}_1 & \mathbf{M}_2 & \mathbf{M}_3 & \dots & \mathbf{M}_n \\ \mathbf{M}_2 & \mathbf{M}_3 & \dots & \dots & \mathbf{0} \\ \mathbf{M}_3 & \vdots & \ddots & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{M}_n & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \end{bmatrix} \right) \begin{pmatrix} \boldsymbol{\phi} \\ \lambda \boldsymbol{\phi} \\ \lambda^2 \boldsymbol{\phi} \\ \vdots \\ \lambda^{n-1} \boldsymbol{\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{pmatrix} \quad (20)$$

as expressed for one eigenpair. In fact, one checks that the first line of equations of the augmented matrix system above corresponds exactly to Eq. (15) when also expressed for one eigenpair. Once an eigenpair containing the indicated augmented eigenvector has been evaluated according to Eq. (20), only the first subvector  $\boldsymbol{\phi}$  is of actual interest. Although mathematically equivalent, there are two practical differences between these formulations: Equation (20) requires a large storage allocation and more computational time to be solved, whereas operations with Eq. (15) are proner to round-off errors owing to the high exponentiation of  $\lambda$ .

Independently from whether arranging the eigenvalue problem as in Eq. (15) or in Eq. (20), convergence with the Jacobi-Davidson algorithm for complex-symmetric matrices, as more generally applied to problems with viscous damping, is demonstrably cubic (Dumont, 2007) – as opposed to quadratic when the matrix system is Hermitian. On the other hand, convergence can only be guaranteed if all generalized mass matrices are positive definite and if the numerical problem is not degraded due to approximating or round-off errors. Moreover, although the algorithm usually starts converging to the lowest eigenvalue (as one starts with zero as the initial guess in the iterative process), there is no mathematical guarantee

that the subsequent eigenvalues will be the next smallest ones (for a problem without damping), so that relevant eigenpairs may be missed in the analysis.

## 5 ON EIGENVALUE DEFLATION METHODS

Several deflation methods for the linear eigenvalue problem

$$(\mathbf{K}_0 - \lambda \mathbf{M}_1) \boldsymbol{\phi} = \mathbf{0}, \quad (21)$$

as expressed in the notation used in this paper, are known in the technical literature (Bunse-Gerstner, 1984; Felippa, 1998; Money and Ye, 2005; Pereira and Rosa, 2011), which are also used to cancel out resonance modes in a dynamics problem (Tisseur and Meerbergen 2001).

For the very classical problem of the identity matrix substituting for the mass matrix in Eq. (21), the Wielandt deflation technique consists in transforming the initial matrix  $\mathbf{K}_0$  into a matrix  $\tilde{\mathbf{K}}_0$  in such a way that a set of already known eigenvalues become very large and, as a result, the corresponding eigenpairs are not likely to be found in the next evaluation steps. A particular case of the Wielandt technique is the Hotelling deflation, in which the problem is modified in such a way that an already known eigenvector will correspond to a zero eigenvalue. This latter technique reduces the matrix rank and may be related to techniques that also reduce the matrix order.

### 5.1 Deflation method proposed by Money and Ye (2005)

The Wielandt deflation technique is applied by Money and Ye (2005) to Eq. (21), thus proposing a modified stiffness matrix to deflate the eigenpair  $(\lambda_i, \boldsymbol{\phi}_i)$

$$\tilde{\mathbf{K}}_0 = \mathbf{K}_0 + (\sigma - \lambda_i) \mathbf{M}_1 \boldsymbol{\phi}_i \boldsymbol{\phi}_i^T \mathbf{M}_1 \quad (22)$$

for a sufficiently large value  $\sigma$  and while keeping  $\tilde{\mathbf{M}}_1 = \mathbf{M}_1$  unaltered.

### 5.2 Some deflation techniques for quadratic and cubic problems

The problem stated in Eq. (15) may also be written as

$$(\mathbf{K}_0 - \lambda \mathbf{M}_1 - \lambda^2 \mathbf{M}_2 - \lambda^3 \mathbf{M}_3 - \dots) \boldsymbol{\phi} = \mathbf{0} \quad (23)$$

Deflation techniques for the quadratic eigenvalue problem are found in the technical literature (Hwang et al., 2003; Chu et al., 2005; Carvalho et al., 2006).

For a quadratic problem, Chu et al. (2005) propose a technique based on the orthogonalization and normalization criteria

$$\boldsymbol{\phi}^T \mathbf{M}_2 \boldsymbol{\phi} = 1 \quad \text{and} \quad \boldsymbol{\phi}^T \mathbf{K}_0 \boldsymbol{\phi} = \lambda \quad (24)$$

Then, although the proposed matrix substitutions preserve the symmetry property of the original Eq. (23), this is not a true or useful solution in the frame of the generalized mode superposition analysis proposed by Dumont (2007) – compare with Eqs. (16) and (17) – and cannot be applied to arrive at the uncoupled system of differential equations of time displayed in Eq. (19).

The method proposed by Hwang et al. (2003) preserves the sparsity structure of the participating matrices, but not the required symmetry and the orthogonality properties of Eqs.

(16) and (17), as required in a dynamics analysis. Hwang et al. (2005) propose a deflation method for cubic problems that can be subjected to the same criticism of the other techniques (Carvalho, 2017).

## 6 A NOVEL, GENERALIZED DEFLATION TECHNIQUE APPLIED TO NONLINEAR PROBLEMS

Given the negative result of our literature survey to find a deflation technique applicable to the generalized, nonlinear eigenvalue problem stated in Eq. (15) and in the frame of a mode superposition analysis, we tried to develop our own procedure (Carvalho, 2017) starting from the proposition by Money and Ye (2005) for the linear problem.

The generalized eigenvalue problem of Eq. (23) is more consistently expressed, according to Dumont (2007), as

$$\left(\mathbf{K}_{(\lambda)} - \lambda \mathbf{M}_{(\lambda)}\right) \boldsymbol{\phi} = \mathbf{0}, \quad (25)$$

where  $\mathbf{K}_{(\lambda)}$  and  $\mathbf{M}_{(\lambda)}$  are the eigenvalue-dependent, nonlinear stiffness and mass matrices

$$\mathbf{K}_{(\lambda)} = \mathbf{K}_0 + \sum_{j=2}^n (j-1) \lambda^j \mathbf{M}_j = \mathbf{K}_0 + \lambda^2 \mathbf{M}_2 + 2\lambda^3 \mathbf{M}_3 + 3\lambda^4 \mathbf{M}_4 + \dots \quad (26)$$

$$\mathbf{M}_{(\lambda)} = \sum_{j=1}^n j \lambda^{j-1} \mathbf{M}_j = \mathbf{M}_1 + 2\lambda \mathbf{M}_2 + 3\lambda^2 \mathbf{M}_3 + 4\lambda^3 \mathbf{M}_4 + \dots \quad (27)$$

These matrices, when further generalized as functions of two different eigenvalues  $\lambda_r$  and  $\lambda_s$ ,

$$\mathbf{K}_{(\lambda_r, \lambda_s)} = \mathbf{K}_0 + \sum_{j=2}^n \sum_{k=1}^{j-1} \lambda_r^k \lambda_s^{j-k} \mathbf{M}_j = \mathbf{K}_0 + \lambda_r \lambda_s \mathbf{M}_2 + (\lambda_r^2 \lambda_s + \lambda_s^2 \lambda_r) \mathbf{M}_3 + \dots \quad (28)$$

$$\mathbf{M}_{(\lambda_r, \lambda_s)} = \sum_{j=1}^n \sum_{k=1}^j \lambda_r^{k-1} \lambda_s^{j-k} \mathbf{M}_j = \mathbf{M}_1 + (\lambda_r + \lambda_s) \mathbf{M}_2 + (\lambda_r^2 + \lambda_r \lambda_s + \lambda_s^2) \mathbf{M}_3 + \dots \quad (29)$$

fulfill the orthogonality and normalization criteria

$$\boldsymbol{\phi}_r^T \mathbf{K}_{(\lambda_r, \lambda_s)} \boldsymbol{\phi}_s = \delta_{rs} \lambda_r \quad (30)$$

$$\boldsymbol{\phi}_r^T \mathbf{M}_{(\lambda_r, \lambda_s)} \boldsymbol{\phi}_s = \delta_{rs}$$

which are also further generalizations of Eqs. (16) and (17) and enter the algorithm for nonlinear eigenvalue problems proposed by Dumont (2007).

Then, in order to deflate an eigenpair  $(\lambda_1, \boldsymbol{\phi})$  in such a way that Eqs. (16) and (17) still hold and the structure of Eq. (15) is preserved, we substitute for matrices  $\mathbf{K}_{(\lambda)}$  e  $\mathbf{M}_{(\lambda)}$  (Carvalho, 2017) as a generalization of Money and Ye's (2005) proposition of Eq. (22):

$$\begin{aligned} \tilde{\mathbf{K}}_{(\lambda)} &= \mathbf{K}_{(\lambda_1, \tilde{\lambda})} + (\sigma - \lambda_1) \mathbf{M}_{(\lambda_1, \tilde{\lambda})} \boldsymbol{\phi} \boldsymbol{\phi}^T \mathbf{M}_{(\lambda_1, \tilde{\lambda})} \\ \tilde{\mathbf{M}}_{(\lambda)} &= \mathbf{M}_{(\lambda_1, \tilde{\lambda})} \end{aligned} \quad (31)$$

where  $\tilde{\lambda}$  is a test eigenvalue whose meaning is to be identified in the following developments. Observe that the expression of  $\tilde{\mathbf{K}}_{(\lambda)}$  contains in principle three different eigenvalues:  $\lambda$  of a general eigenpair  $(\lambda, \phi)$ ,  $\lambda_1$  of the eigenpair  $(\lambda_1, \phi_1)$  to be deflated and a test eigenvalue  $\tilde{\lambda}$ . The deflated eigenvalue equation reads

$$\left(\tilde{\mathbf{K}}_{(\lambda)} - \lambda \tilde{\mathbf{M}}_{(\lambda)}\right) \phi = \mathbf{0} \quad (32)$$

and two cases must be considered for the assessment of  $\tilde{\lambda}$  in Eqs. (31) and (32).

**First case:** Equation (32) is tested for  $\tilde{\lambda} = \lambda_1$ , that is, for the eigenpair  $(\tilde{\lambda}, \phi) \equiv (\lambda_1, \phi_1)$ . Then, this equation is expressed from Eq. (31) as

$$\left[\mathbf{K}_{(\lambda_1, \lambda_1)} - \lambda \mathbf{M}_{(\lambda_1, \lambda_1)} + (\sigma - \lambda_1) \mathbf{M}_{(\lambda_1, \lambda_1)} \phi_1 \phi_1^T \mathbf{M}_{(\lambda_1, \lambda_1)}\right] \phi_1 = \mathbf{0}, \quad (33)$$

which, according to Eq. (30) and the notation used in Eqs. (26) and (27), is the same as

$$\left[\mathbf{K}_{(\lambda_1)} - \lambda \mathbf{M}_{(\lambda_1)} + (\sigma - \lambda_1) \mathbf{M}_{(\lambda_1)}\right] \phi_1 = \mathbf{0} \quad (34)$$

and holds only if  $\lambda = \sigma$  (two terms cancel out), which means that the eigenpair  $(\lambda_1, \phi_1)$  has been in fact deflected to the eigenpair  $(\sigma, \phi_1)$ , as required.

**Second case:** Equation (32) is tested for  $\tilde{\lambda} = \lambda$ , that is, for the generic eigenpair  $(\lambda, \phi) \neq (\lambda_1, \phi_1)$ , which is actually the intended practical application. Now, this equation reads

$$\left[\mathbf{K}_{(\lambda_1, \lambda)} - \lambda \mathbf{M}_{(\lambda_1, \lambda)} + (\sigma - \lambda_1) \mathbf{M}_{(\lambda_1, \lambda)} \phi_1 \phi_1^T \mathbf{M}_{(\lambda_1, \lambda)}\right] \phi = \mathbf{0}, \quad (35)$$

which, according to Eq. (30), results into

$$\left(\mathbf{K}_{(\lambda_1, \lambda)} - \lambda \mathbf{M}_{(\lambda_1, \lambda)}\right) \phi = \mathbf{0}. \quad (36)$$

This equation has the same pattern of Eq. (32) but turns out to be exactly the original Eq. (25), as the terms in  $\lambda_1$  cancel out.

This two-case study attests the consistency of the deflated expression of Eq. (32) using the generalized substitution given in Eq. (31).

Since one is actually interested in using the deflated Eq. (32) applied to the case when  $\tilde{\lambda} = \lambda$ , Eq. (35) must be expressed in the general pattern of Eq. (23)

$$\left(\tilde{\mathbf{K}}_0 - \lambda \tilde{\mathbf{M}}_1 - \lambda^2 \tilde{\mathbf{M}}_2 - \lambda^3 \tilde{\mathbf{M}}_3 - \dots\right) \phi = \mathbf{0} \quad (37)$$

where individually deflated matrices  $\tilde{\mathbf{K}}_0$  and  $\tilde{\mathbf{M}}_i$  are identified as functions of  $\sigma$  and  $\lambda_1$  from Eq. (35) written in terms of the expansions in Eqs. (28) and (29)

$$\left[\mathbf{K}_0 + \sum_{j=2}^n \sum_{k=1}^{j-1} \lambda^k \lambda_1^{j-k} \mathbf{M}_j - \lambda \sum_{j=1}^n \sum_{k=1}^j \lambda^{k-1} \lambda_1^{j-k} \mathbf{M}_j + (\sigma - \lambda_1) \left(\sum_{j=1}^n \sum_{k=1}^j \lambda^{k-1} \lambda_1^{j-k} \mathbf{M}_j\right) \phi_1 \phi_1^T \left(\sum_{j=1}^n \sum_{k=1}^j \lambda^{k-1} \lambda_1^{j-k} \mathbf{M}_j\right)\right] \phi = \mathbf{0}, \quad (38)$$

from which results after a tedious manipulation



$$\tilde{\mathbf{K}}_0 = \mathbf{K}_0 + (\sigma - \lambda_1) \sum_{j=1}^n \sum_{k=1}^n \lambda_1^{j+k-2} \mathbf{M}_j \boldsymbol{\phi} \boldsymbol{\phi}^T \mathbf{M}_k \quad (39)$$

$$\tilde{\mathbf{M}}_i = \mathbf{M}_i - (\sigma - \lambda_1) \sum_{k=1}^{i+1} \left[ \left( \sum_{j=k}^n \lambda_1^{j-k} \mathbf{M}_j \right) \boldsymbol{\phi} \boldsymbol{\phi}^T \left( \sum_{j=i-k+2}^n \lambda_1^{j+k-i-2} \mathbf{M}_j \right) \right], \quad i = 1, \dots, n \quad (40)$$

Equation (38) is shown below for an expansion up to  $\lambda^3$  in Eq. (37), which include the lower-order expansion terms for  $\lambda$  and  $\lambda^2$  as truncations marked with (red) vertical lines:

$$\tilde{\mathbf{K}}_0 = \mathbf{K}_0 + (\sigma - \lambda_1) \left[ \mathbf{M}_1 \boldsymbol{\phi} \boldsymbol{\phi}^T \mathbf{M}_1 \Big|^{[1]} + \lambda_1 (\mathbf{M}_1 \boldsymbol{\phi} \boldsymbol{\phi}^T \mathbf{M}_2 + \mathbf{M}_2 \boldsymbol{\phi} \boldsymbol{\phi}^T \mathbf{M}_1) + \lambda_1^2 (\mathbf{M}_2 \boldsymbol{\phi} \boldsymbol{\phi}^T \mathbf{M}_2 \Big|^{[2]} + \mathbf{M}_1 \boldsymbol{\phi} \boldsymbol{\phi}^T \mathbf{M}_3 + \mathbf{M}_3 \boldsymbol{\phi} \boldsymbol{\phi}^T \mathbf{M}_1) + \lambda_1^3 (\mathbf{M}_2 \boldsymbol{\phi} \boldsymbol{\phi}^T \mathbf{M}_3 + \mathbf{M}_3 \boldsymbol{\phi} \boldsymbol{\phi}^T \mathbf{M}_2) + \lambda_1^4 \mathbf{M}_3 \boldsymbol{\phi} \boldsymbol{\phi}^T \mathbf{M}_3 \right] \quad (41)$$

$$\begin{aligned} \tilde{\mathbf{M}}_1 &= \mathbf{M}_1 \Big|^{[1]} - (\sigma - \lambda_1) \left[ \mathbf{M}_1 \boldsymbol{\phi} \boldsymbol{\phi}^T \mathbf{M}_2 + \mathbf{M}_2 \boldsymbol{\phi} \boldsymbol{\phi}^T \mathbf{M}_1 + \lambda_1 \left( 2\mathbf{M}_2 \boldsymbol{\phi} \boldsymbol{\phi}^T \mathbf{M}_2 \Big|^{[2]} + \mathbf{M}_1 \boldsymbol{\phi} \boldsymbol{\phi}^T \mathbf{M}_3 \right. \right. \\ &\quad \left. \left. + \mathbf{M}_3 \boldsymbol{\phi} \boldsymbol{\phi}^T \mathbf{M}_1 \right) + 2\lambda_1^2 (\mathbf{M}_2 \boldsymbol{\phi} \boldsymbol{\phi}^T \mathbf{M}_3 + \mathbf{M}_3 \boldsymbol{\phi} \boldsymbol{\phi}^T \mathbf{M}_2) + 2\lambda_1^3 \mathbf{M}_3 \boldsymbol{\phi} \boldsymbol{\phi}^T \mathbf{M}_3 \right] \\ \tilde{\mathbf{M}}_2 &= \mathbf{M}_2 - (\sigma - \lambda_1) \left[ \mathbf{M}_2 \boldsymbol{\phi} \boldsymbol{\phi}^T \mathbf{M}_2 \Big|^{[2]} + \mathbf{M}_1 \boldsymbol{\phi} \boldsymbol{\phi}^T \mathbf{M}_3 + \mathbf{M}_3 \boldsymbol{\phi} \boldsymbol{\phi}^T \mathbf{M}_1 \right. \\ &\quad \left. + 2\lambda_1 (\mathbf{M}_2 \boldsymbol{\phi} \boldsymbol{\phi}^T \mathbf{M}_3 + \mathbf{M}_3 \boldsymbol{\phi} \boldsymbol{\phi}^T \mathbf{M}_2) + 3\lambda_1^2 \mathbf{M}_3 \boldsymbol{\phi} \boldsymbol{\phi}^T \mathbf{M}_3 \right] \\ \tilde{\mathbf{M}}_3 &= \mathbf{M}_3 - (\sigma - \lambda_1) \left[ \mathbf{M}_2 \boldsymbol{\phi} \boldsymbol{\phi}^T \mathbf{M}_3 + \mathbf{M}_3 \boldsymbol{\phi} \boldsymbol{\phi}^T \mathbf{M}_2 + 2\lambda_1 \mathbf{M}_3 \boldsymbol{\phi} \boldsymbol{\phi}^T \mathbf{M}_3 \right] \\ \tilde{\mathbf{M}}_4 &= -(\sigma - \lambda_1) \mathbf{M}_3 \boldsymbol{\phi} \boldsymbol{\phi}^T \mathbf{M}_3 \end{aligned} \quad (42)$$

In the above cubic expansion a fourth-order term arises – as the product of two second order terms in Eq. (35). Then, in such a case and in all cases for nonlinearity higher than two one must truncate the terms of too higher an order in Eq. (38), as observed by comparing Eqs. (23) and (37) – if such a generalized algorithm is to be applied after all – and as already considered in the compact expression of Eq. (40). Although this explicitly incurs in an approximation, such a truncation is already implicit in the starting power series of Eq. (14). The application of this deflation algorithm to cubic and higher order eigenvalue problems is being presently numerically implemented and assessed for accuracy and feasibility (Carvalho, 2017).

Observe that, once all deflated matrices for the eigenvalue pair  $(\lambda_1, \boldsymbol{\phi})$  have been evaluated, the new nonlinear eigenvalue problem expressed in Eq. (37) is to be solved according to the developments of Section 4 (Dumont, 2007) and, if required, a new deflation can be carried out to exactly as described above, thus considering the deflated matrices of Eqs. (39) and (40) as the entry matrices of Eq. (23).

## 7 SOME NUMERICAL STUDIES

### 7.1 Fixed-free bar

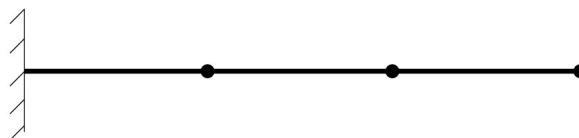
Figure 1 illustrates a fixed-free bar, as explored by Dumont (2007), whose eigenfrequencies will be evaluated in the present paper for a discretization with 30 truss elements and 30 axial degrees of freedom. The relevant properties of this bar are, in consistent units, total length  $\ell = 3$ , elasticity modulus  $E = 100$ , specific mass density  $\rho = 1$  and

arbitrary, but constant cross-area  $A$ , besides assuming absence of damping. The analytical eigenvalues related to the fixed-free bar problem are

$$\lambda_j \equiv \omega_j^2 = \left( \frac{(2j-1)\pi c}{2\ell} \right)^2, \quad j = 1, \dots, 30 \quad (43)$$

where  $\omega_j$  are the eigenfrequencies and  $c = \sqrt{E/\rho}$  is the wave propagation velocity through the elastic medium. This is an extremely simple problem as all mass matrices can be analytically generated with as high a precision as wanted and also with the guarantee that they are all positive definite – and three-diagonal, see Dumont (2007) for the complete problem description including damping. In the present application, all numerical evaluations are carried out with double precision and the convergence tolerance is set equal to  $10^{-10}$ . A maximum number of 50 iterations is considered in the analysis using a code developed by Dumont (2007).

**A linear eigenvalue assessment.** The linear deflation technique proposed by Money and Ye (2005) – a particular case of the developments of Section 6 – is used in a first assessment for an analysis with up to four mass matrices, but for a linear setup in terms of augmented matrices, as given in Eq. (20).



**Figure 1. Scheme of a fixed-free bar as modeled with three truss elements (Dumont, 2007), to be discretized in the present analysis with 30 elements**

Figure 2 displays the eigenfrequencies obtained for the eigenvalue procedure with and without using deflation after each evaluation, as compared to the target results of Eq. (43). Owing to the positive-definiteness of all matrices as well as to their sparsity no round-off error could be detected and all results obtained using deflation present the same accuracy of the corresponding results without deflation within the assumed tolerance threshold. Moreover, all eigenvalues could be obtained sequentially from the smallest to the largest either with or without deflation. The remarkable accuracy improvement obtained with four matrices – as opposed to the classical results using one mass matrix – can be better observed in the error graphic of Fig. 3. Owing to the numerical discretization procedure of the problem the higher-order eigenvalues can only be obtained with decreasing accuracy.

**A nonlinear eigenvalue assessment.** The same fixed-free bar problem of the previous analysis is now numerically simulated in the frame of a quadratic, nonlinear eigenvalue problem, thus considering two mass matrices (MM). The generalized deflation procedure proposed in Section 6 is applied. The results with and without deflation are shown in Fig. 4 together with the target, analytical values of Eq. (43). The relative errors as compared to the analytical results are shown in Fig. 5. They are exactly of the same order of the errors related to the analysis with augmented matrices displayed in Fig. 3 for two mass matrices.

There is no sensible difference of results between the two procedures and all eigenvalues could be evaluated sequentially and with the same accuracy within the proposed threshold. However, it is worth observing that, while all eigenvalues in the procedure without deflation could be obtained sequentially, only the first two eigenvalues in the procedure with deflation were obtained in increasing order of magnitude.

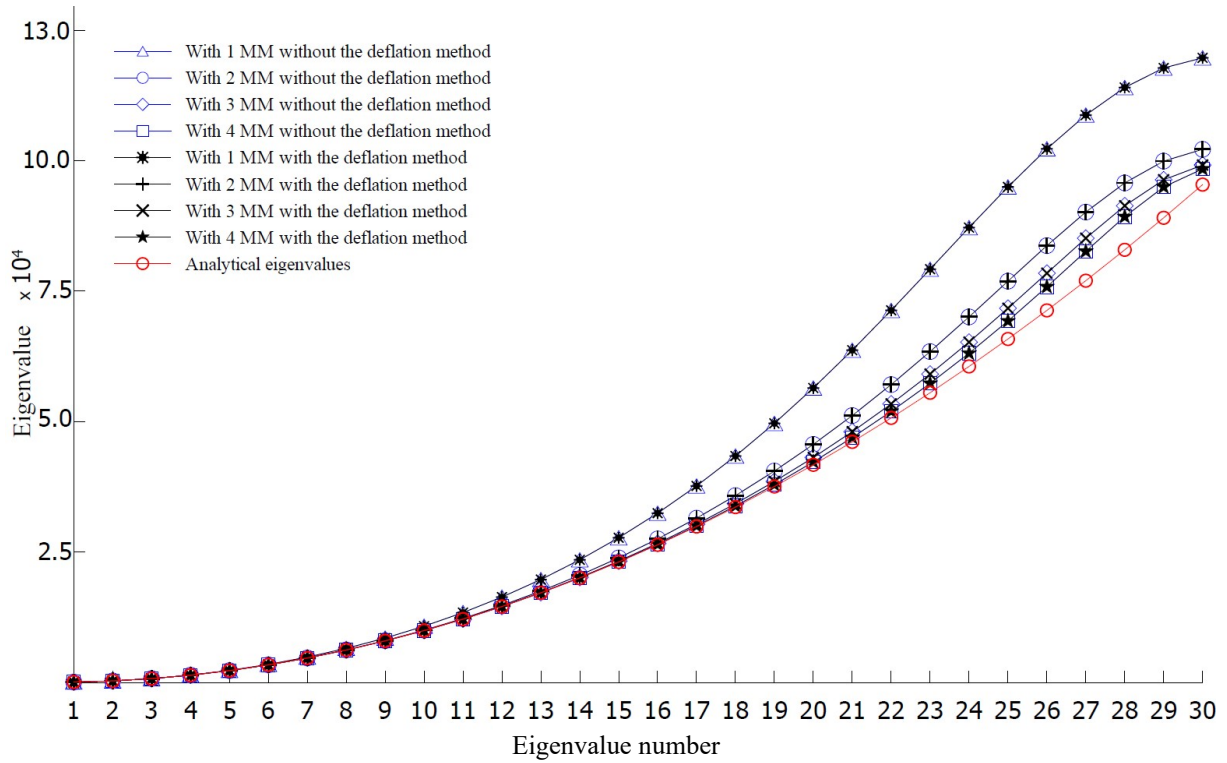


Figure 2. Eigenvalue results for the problem of Fig. 1 using the augmented linear setup of Eq. (20)

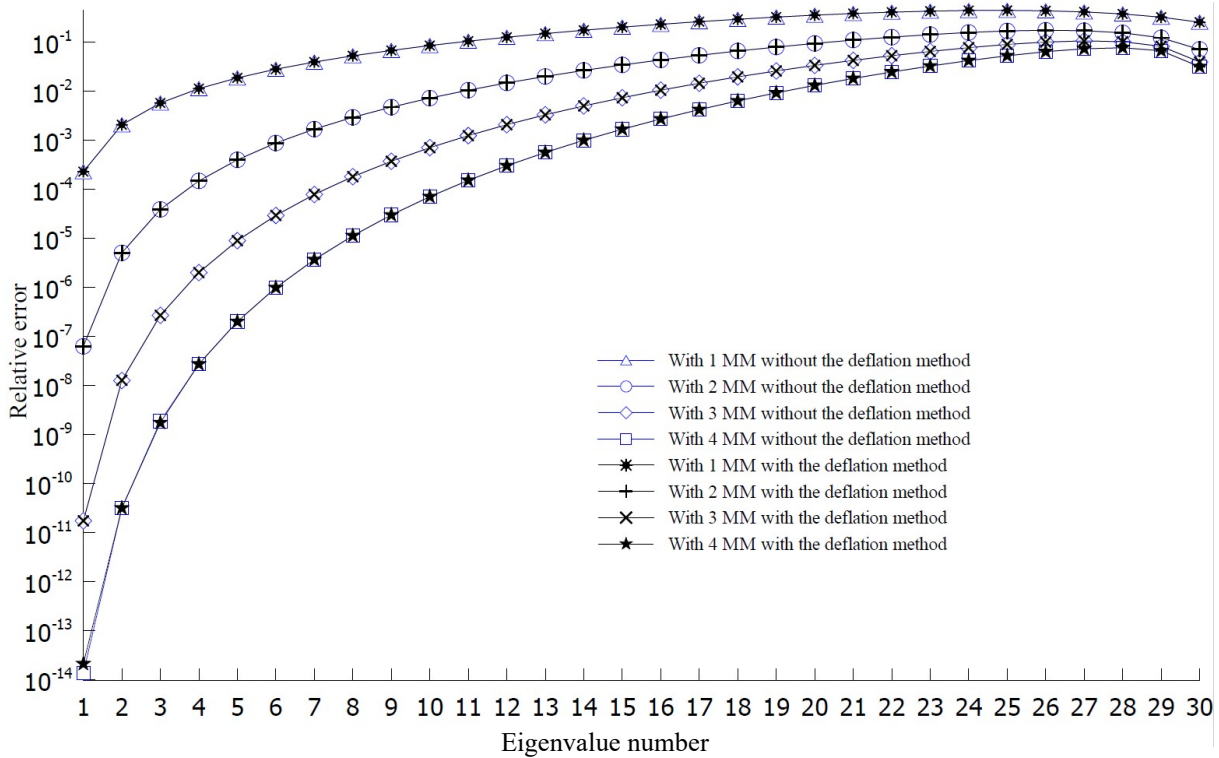


Figure 3. Accuracy assessment of the eigenvalue results of Fig. 2 for the problem of Fig. 1

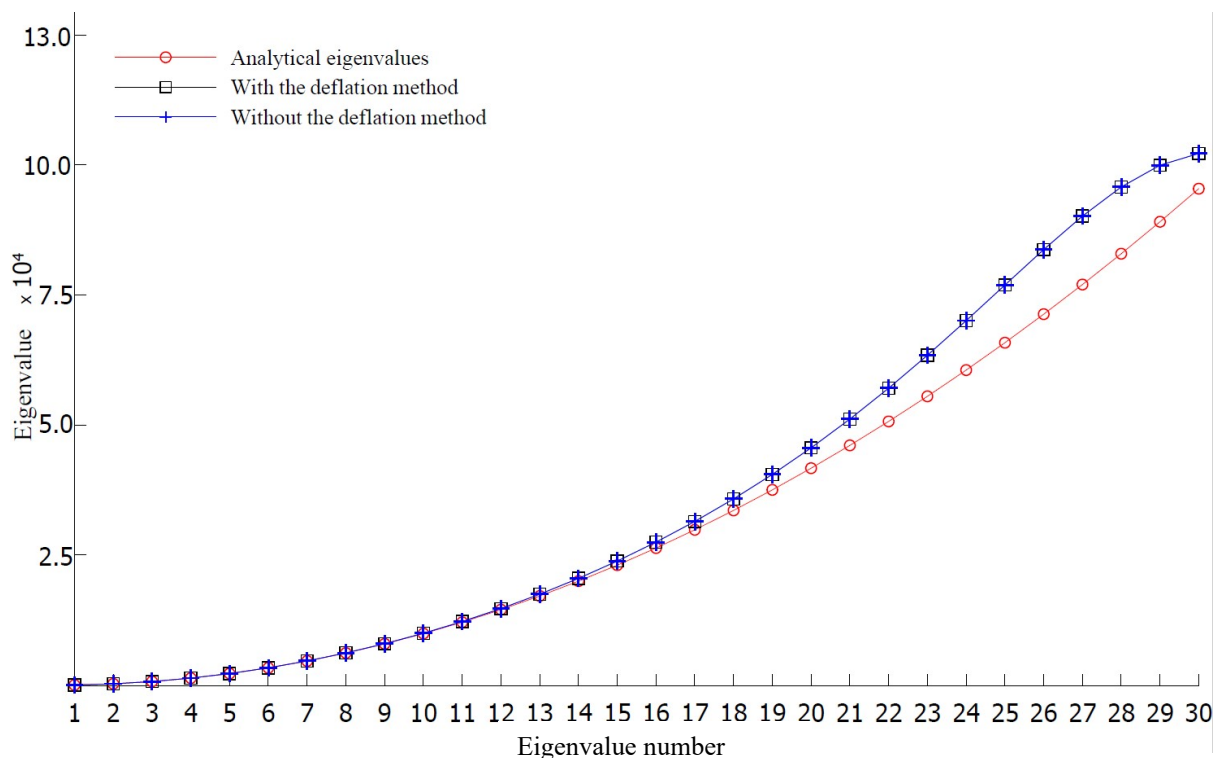


Figure 4. Nonlinear eigenvalue results for the problem of Fig. 1 using two mass matrices

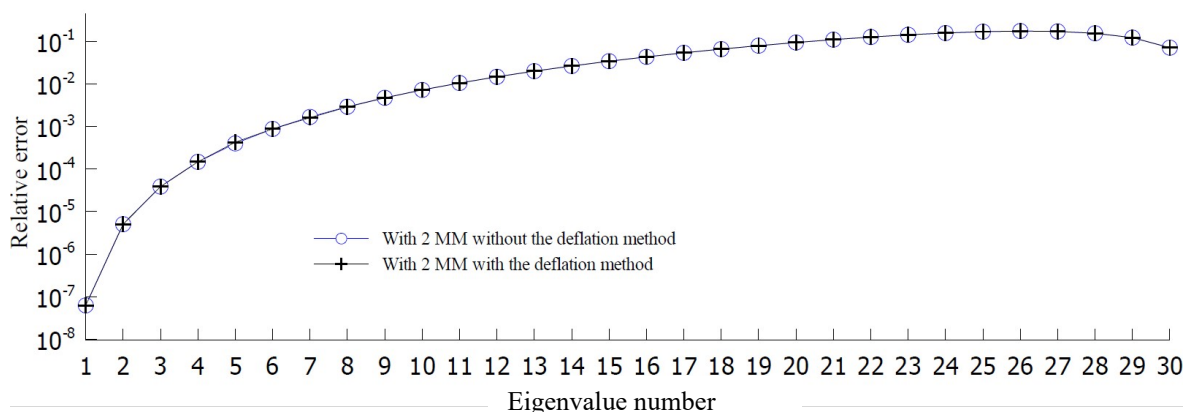


Figure 5. Accuracy assessment of the nonlinear eigenvalue results of Fig. 4 for the problem of Fig. 1

## 7.2 A cut-out test

The same fixed-free bar of the previous example is now analyzed in a very challenging framework. It is adapted from a simulation prepared by Chaves (2003) and also presented by Dumont and Chaves (2003), where a full time-dependent analysis is carried out using the advanced mode superposition procedure outlined in Section 4 with the nonlinear eigenvalue problem solved in terms of a linear augmented system. The present results are only concerned with the eigenvalue analysis. Figure 6 displays an irregular cut-out of a  $50 \times 35 \times 1 m^3$  solid, fixed-free elastic body (wide bar element) submitted to a pulsating load of total intensity  $P(t) = -10^8 \sin(1.35\omega_0 t)$ , where  $\omega_0 = \pi c / (2\ell)$ , exactly as already introduced in Eq. (43) and

further analyzed by Carvalho (2017). The basic properties for this problem are length  $\ell = 50\text{ m}$ , elasticity modulus  $E = 210\text{ GPa}$ , Poisson's ratio  $\nu = 0$  and inertia  $\rho = 7850\text{ Kg/m}^3$ .

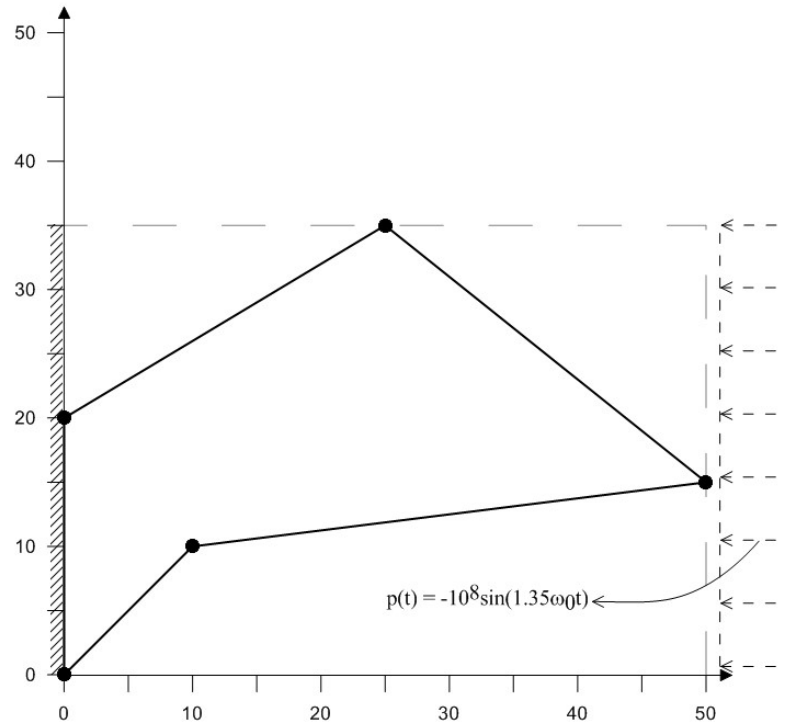


Figure 6. Irregularly shaped body cut out of a truss element – adapted from Dumont and Chaves (2003) and Chaves (2003)

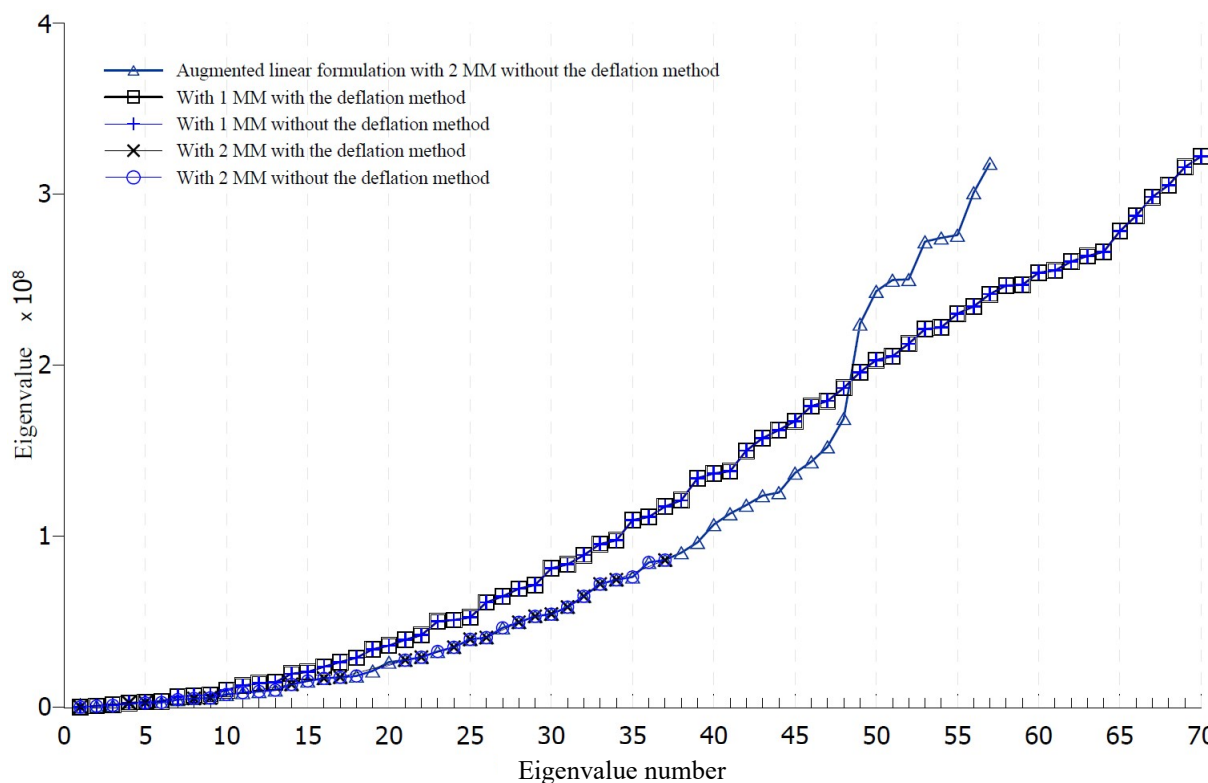
This irregular patch is discretized with 90 (almost) equally spaced linear boundary elements. Owing to the one-dimensional character of the excitation (Poisson's ratio  $\nu = 0$ ), this problem can be solved in the frame of the potential theory. After cancelling 19 rows and columns of all matrices corresponding to the prescribed, Dirichlet zero displacements on the left edge of the structure the problem reduces to order 71.

Figure 7 presents the eigenvalues obtained using one and two mass matrices (MM) with or without deflation as well as a case of two mass matrices in the frame of a linear, augmented setup.

For the case of one mass matrix all 71 eigenvalues could be evaluated both with and without deflation (cases marked with  $\square$  and  $+$  in the graphics, respectively) with coincidental results within the assumed error tolerance of  $10^{-10}$ . The nonlinear algorithm for two mass matrices without deflation ( $\circ$ ) detected 34 eigenvalues whereas only 21 eigenvalues could be found by applying deflation each time ( $\times$ ). The linear, augmented formulation for two matrices and without deflation, as used by Dumont and Chaves (2003), arrived at 57 eigenvalues ( $\Delta$ ). In neither of the five cases described could the eigenvalues be obtained in sequentially increasing order of size, although the first eigenvalue has consistently been the first to be obtained, which is likely to occur, but not a mathematical certainty (Dumont, 2007).

The results in Fig. 7 are ordered by size. In the case of two mass matrices, this ordering has become sometimes tricky and one can observe several gaps in the sequence. The results for larger eigenvalues obtained with the linear, augmented algorithm are shown as if they are larger than in the case with one mass matrix. This is only a matter of arrangement, as it is

generally not possible to establish a correspondence between the values with one and two mass matrices, as the larger eigenvalues differ very much in size. The results with two mass matrices are likely to be the improved ones although several eigenvalues could not be found. Nevertheless, the time-domain description of a problem with more mass matrices seems to always represent an improvement in spite of some missing eigenvalues, as already tested by Dumont and Chaves (2003).



**Figure 7. Comparison of evaluated eigenvalues for the problem of Fig. 6 with and without deflation**

The undesirable fact of missing eigenvalues – and mainly the lack of control of their size in their evaluation sequence – is one of the motivations of the present work. The clean results of the first example could not be reproduced in the present numerical simulations, as now one is dealing with full matrices, which, although (somehow forcedly) symmetric, are not positive definite owing to the problem's boundary approximation (which does not occur with truss and beam elements) as well as to round-off errors. The above described example shows that the applied deflation technique may be of no advantage at all (only 21 eigenvalues detected).

As a matter of fact, Fig. 8 displays the number of iterations required in the generalized Jacobi-Davidson algorithm proposed by Dumont (2007) to attain convergence for each of the eigenvalues found. The algorithm with deflation in general requires more iterations and is also more time consuming.

However, there is still room for improvements such as in the choice of the large value  $\sigma$  of the deflated eigenvalue used in Section 6. A possibility could be the use of  $\sigma = 0$ , which can be seamlessly applied in the frame of the developed nonlinear deflation technique and would coincide with a generalized Hotelling deflation.



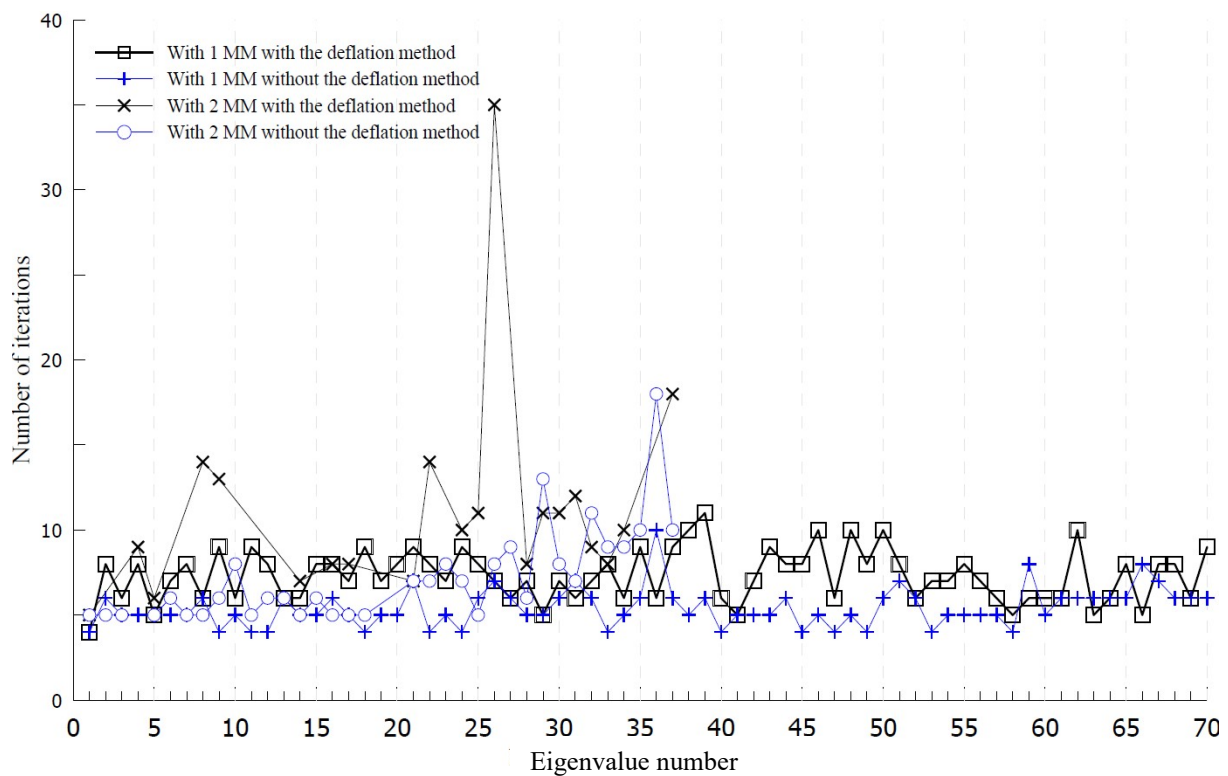


Figure 8. Number of iterations required for convergence in four of the cases of Fig. 7

## 8 STUDY IN PROGRESS: CORRECT NUMERICAL EVALUATION OF NEARLY SINGULAR INTEGRALS

The numerical experimentations carried out for the examples of Sections 7.1 and 7.2 show that, although the nonlinear eigenvalue analysis proposed by Dumont (2007) has a sound mathematical basis, it may not be satisfactorily applicable to problems with generalized mass matrices whose positive-definiteness cannot be demonstrated. Large and fully populated – but positive definite – matrices can in principle be adequately treated with controllable round-off errors provided that a sufficiently large computational precision is used, such as in the frame of the software implementations by Advanpix ([www.advanpix.com](http://www.advanpix.com)). However, it is essential, if the generalized mode superposition analysis proposed by Dumont (2007) is to be of practical use, that the participating mass matrices have their intrinsic linear algebra properties preserved. The most important step in this direction is the implementation of a highly accurate numerical integration technique.

The issue of carrying out an accurate evaluation of the improper and singular integrals inherent to the boundary element methods has been satisfactorily solved a long time ago. The still not entirely solved issue concerns the case of nearly singular integrals. Several schemes have been proposed, such as in terms of interval subdivision (Lachat and Watson, 1976; Yang and Atkinson, 1993), regularization (Hayami and Brebbia, 1988; Huang and Cruse, 1993) or coordinate transformation (Telles, 1987; Zhang et al. 2009; Xie et al. 2011; Zhang et al. 2013). The most efficient of such techniques has been proposed by Dumont (1994) and was further developed by Dumont and Noronha (1998) – see also Noronha (1998). This latter technique has a strong mathematical basis, namely the fact that a nearly singular integral has

complex singularity poles that need to be taken into account whether or not the underlying fundamental solution is a function of real variables. Thus proceeding, quasi singular integrals can be numerical evaluated as accurately as desired and at a low computational cost, although the code becomes more involved.

This procedure is being presently implemented by the first author (Carvalho, 2017) for the generation of generalized mass matrices. There will remain a lack of positive-definiteness related to the unavoidable boundary approximations, but at least a convergence analysis will be made possible.

## CONCLUSIONS

The generalized modal analysis of a time-dependent potential or elasticity problem is briefly presented in this paper, which mainly deals with the issues of solving a nonlinear eigenvalue problem in terms of mass matrices that should be – but in the real world are not – positive definite. The eventual solution of the nonlinear eigenvalue problem by making use of deflation is discussed and a formulation is eventually proposed and assessed by means of a few numerical examples. Owing to its novelty, this formulation is per se a valuable mathematical contribution. Moreover, it may be premature to conclude that the proposed deflation technique does not contribute to the solution of a problem whose matrices present some deterioration of their linear algebra properties, as for instance the adequate numerical assignment of the deflated eigenvalue still deserves a careful investigation. A code that works with multiple precision must be tested. Further developments – such as the accurate evaluation of nearly singular integrals and as a result the better approximation of the evaluated matrices to positive definite ones – may also lead to the improvement of the nonlinear eigenvalue analysis. For the sake of simplicity, the studies carried out so far do not consider viscous damping. This can be seamlessly taken into account as its general mathematical outline is already given (Dumont, 2007).

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