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FAST BOUNDARY ELEMENT SIMULATIONS FOR COMPLEX GEOMETRY BIDIMENSIONAL ENCLOSURES

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Abstract. *In this paper acoustic enclosures with numerous holes presenting complex geometry are investigated using an accelerated two-dimensional boundary element method. The results are obtained using the two-dimensional boundary element method (BEM) accelerated by the adaptive cross-approximation (ACA) and generalized minimum residual (GMRES) solver. A numerical analysis is carried out to assess the accuracy and efficiency of the method compared with the traditional boundary element method.*

Keywords: *Acoustics, boundary element method, adaptive cross-approximation, numerical methods*

1 INTRODUCTION

The Boundary Element Method (BEM) is a powerful numerical method for solving acoustic problems governed by the Helmholtz equation. Boundary Element Method models tend to be competitive mainly due to ease of discretization. Using the Boundary Element Method, only the boundary of the model needs to be discretized, effectively eliminating one dimension of the problem. In acoustic scattering problems, as with any problem with infinite domains, the Boundary Element Method presents an even more appealing facet: there is no need to discretize the whole domain. The mathematical internal structure of the method already satisfies the Sommerfeld radiation condition, scatterers need only to have their boundary meshed, rather than the complete domain. Another advantage of the BEM is the evaluation of quantities outside of the boundary of the problem. Every quantity is evaluated in the boundary of the problem in the BEM, and quantities can be evaluated outside the boundary as post-processing and can be located anywhere inside the domain without any restrictions. In external problems, this means that any point from the boundary of the scatterer arbitrarly away from it can be evaluated.

Historically, the BEM haven't obtained much popularity for a very simple reason: it generates fully populated non-symmetric matrices. The linear system which describes the problem can be easily solved using traditional methods, but the computational cost of storing the matrices is of order $\mathrm{O}(N^2)$ and it needs $\mathrm{O}(N^3)$ operations to solve the linear system by direct methods and $\mathrm{O}(N^2)$ for iterative methods. It is not very competitive with other numerical methods in its most basic form. For this reason, BEM models have been limited to a few thousands of degrees of freedom for almost twenty years. Some acceleration methods have been proposed to circumvent this problem. The fast multipole method was the first one to be fully formulated, but its implementation is complex and laborious. There are many BEM formulations that use the fast multi-pole method to improve the efficiency of the models, but this inflexibility is a major concern when considering improvements and variations on the formulation.

The adaptive cross-approximation (ACA) is another proposed solution to accelerate the BEM. It consists of a methodology to approximate the dense BEM influence matrix using low-rank matrices, reducing both the processing and memory requirements for large scale problems. This low-rank representation is obtained using a tree like structure which divides the domain in separate blocks using geometric criteria. The blocks can be approximated by low-rank matrices if the behaviour of the block is suitable. This suitability is verified recursevely and if the problem does not allow for any block to be approximated by the ACA, the influence matrix is not approximated and the full problem is solved.

2 THE BOUNDARY ELEMENT METHOD

The formulation used in this work is the direct boundary element method for the Helmholtz equation, described in more details in (DOMINGUEZ, 1993), (WROBEL, 2001) and (KIRKUP, 2007).

The propagation of acoustic waves through a fluid medium Ω is described by the wave equation. When the motion is assumed to be time-harmonic, the wave equation reduces to the Helmholtz equation and take the form shown in Eq. (1).

$$\nabla^2\phi + k^2\phi = 0 \tag{1}$$

where ϕ is a reduced velocity potential, $k = \omega/c$ is the wave number, c is the speed of sound and ω is the angular frequency. The velocity potential ϕ is related to the acoustic pressure p as shown in Eq. (2).

$$p = -i\rho\omega\phi \tag{2}$$

where ρ is the density of the fluid medium Ω.

The boundary integral equation for the problem can be found by starting from Green's second identity written in Eq. (3).

$$\int_\Omega \left(\phi\nabla^2\phi^* - \phi^*\nabla^2\phi\right) d\Omega = \int_\Gamma \left(\phi\frac{\partial\phi^*}{\partial n} - \phi^*\frac{\partial\phi}{\partial n}\right) d\Gamma \tag{3}$$

where ϕ^* is a weight function and Γ is the boundary of domain Ω. Substituting the Laplacian $\nabla^2\phi$ in Eq. (1) for $-k^2\phi$ from Eq. (3) gives

$$\int_\Omega \phi\left(\nabla^2\phi^* + k^2\phi^*\right) d\Omega = \int_\Gamma \left(\phi\frac{\partial\phi^*}{\partial n} - \phi^*\frac{\partial\phi}{\partial n}\right) d\Gamma \tag{4}$$

If the weight function ϕ^* satisfies Eq. (5), then the function is said to be a fundamental solution of Eq. (1) and it corresponds to the field generated by a unit concentrated harmonic source.

$$\nabla^2\phi^*(X', x) + k^2\phi^*(X', x) = -\delta(X', x) \tag{5}$$

The fundamental solution for the bidimensional Helmholtz equation is shown in Eqs. 6 and 7.

$$\phi^* = \frac{1}{2\pi}K_0\left(\frac{i\omega r}{c}\right) \tag{6}$$

$$\frac{\partial\phi^*}{\partial n} = \frac{\omega}{2\pi c}\frac{\partial r}{\partial n}K_1\left(\frac{i\omega r}{c}\right) \tag{7}$$

where K_0 and K_1 are the modified Bessel function of zero and first order and r is the distance between the field and source points.

The integral equation for the boundary element method, obtained by applying the Dirac's delta properties and the Green's Theorem in Eq. (4) is given by Eq. (8).

$$c(x')\phi(x') = \int_\Gamma \frac{\partial \phi(x)}{\partial n}\phi^*(x', x)d\Gamma - \int_\Gamma \phi(x)\frac{\partial \phi^*(x', x)}{\partial n}d\Gamma \tag{8}$$

where $r = |X' - x|$ is the distance between the source point X' and the field point x. Taking X' to the boundary Γ, in view of the behaviour of the fundamental solution when $X' \to x'$, $x' \in \Gamma$ produces Eq. (8).

Two types of approximations are required for the application of the method: the first is geometrical and involves a subdivision of boundary Γ in N small segments, such that $\Gamma \approx \sum_{i=1}^{N} \Gamma_i$; the second is an approximation of the variation of the velocity potential and its normal derivative within each element. The simplest possible approximation is a constant one, which assumes that ϕ and $\partial\phi/\partial n$ are constant within each element and equal to their value at the midpoint. Introducing this approximations, one obtains Eq. (9).

$$c_j\phi_j = \sum_{i=1}^{N} \frac{\partial \phi_i}{\partial n}\int_{\Gamma_i} \phi^*(x', x)d\Gamma - \sum_{i=1}^{N} \phi_i \int_{\Gamma_i} \frac{\partial \phi^*(x', x)}{\partial n}d\Gamma \tag{9}$$

where ϕ_i and $\partial\phi_i/\partial n$ are the values of ϕ and $\partial\phi/\partial n$ at the node located in the midpoint of element i. In the case of constant elements, the number of nodes is equal to the number of elements.

Equation (9) takes the following form

$$\sum_{i=1}^{N} H_{ij}\phi_i = \sum_{i=1}^{N} G_{ij}\frac{\partial \phi_i}{\partial n} \tag{10}$$

Using a collocation technique to all nodal points along the boundary gives a system of equations which can be written in matrix form as shown in Eq. (11).

$$[H]\{\phi\} = [G]\{q\} \tag{11}$$

where $\{\phi\}$ and $\{q\}$ are vectors containing the nodal values of the potential and its normal derivative; $[H]$ and $[G]$ are square $N \times N$ matrices of influence coefficients. The diagonal term $[H_{ii}]$ is integrated analytically and its value corresponds to $[H_{ii}] = 1/2$. The term $[G_{ii}]$ is integrated numerically, with the difference that the unitary normal vector is substituted by a null vector.

Once the boundary conditions of the problem are applied, the problem stated in Eq. (11) can be rearranged in the form shown in Eq. (12).

$$[A]\{x\} = \{b\} \tag{12}$$

where $[A]$ is a matrix containing all knowns, $\{x\}$ is a vector containing all unknowns of the problem and $\{b\}$ is the 'load' vector. This system is solved using a standard direct solver. A reverse arrangement is used to obtain the original system $[H]\{p\} = [G]\{q\}$.

3 THE ADAPTIVE CROSS APPROXIMATION

The adaptive cross approximation (ACA) will be briefly discussed in this section. The first step when applying the ACA methodology in a BEM model is to divide the domain into blocks, which is done according to an hierarchical matrix division, shown in sub-section 3.1. Then, these blocks are approximated by low-rank matrices, which is shown in sub-section 3.2

The ACA BEM is a subject of study of mathematics and engineering departments alike, with many efforts directed to reduce the processing demand of the BEM. Many different differential problems are being handled using the ACA BEM, including problems in the areas of acoustics, heat transfer, electromagnetism, particle behaviour, phononic behaviour and purely mathematical problems (LIU; NISHIMURA, 2006), (KURZ; RAIN; RJASANOW, 2002), (ROKHLIN, 1985), (FREDERIX; BAREL, 2010) and (BEBENDORF, 2008).

3.1 Hierarchical clustering

The ACA relies on the possibility of approximating the influence matrix into a low-rank representation, but this is mainly not the case when we consider the inner structure of the matrix. First, it is necessary to divide the matrix into blocks, which will then be tested for approximation. The methodology used for this division used in this work is called hierarchical clustering.

Low rank approximations can be obtained for low rank matrices, significantly reducing the computational cost of storing and manipulating large matrices. But influence matrices obtained from integral equations have no explicit structure in general. However, low rank blocks can be observed in the BEM influence matrices considering that the integrals of contiguous elements due to a single collocation point are almost identical, especially for high density meshes (BRANCATI, 2010). Some blocks of the influence matrix may present a low rank, which means these blocks are suitable for approximation.

In this work, the influence matrix will be subdivided into two blocks *low rank* and *full rank* blocks. The subdivision is based upon a geometrical criterion of the discretised mesh. This division is done by storing the indices of contiguous nodes and elements in a *cluster tree*, creating a basis for the block subdivision and for building the *block tree*. Once this procedure is finished, low rank blocks are determined using an *admission criterion*, based on geometrical consideration.

The domain will be divided into a cluster tree, which is created by subdividing the matrix which contains the geometric position of the physical points into either branches or leaves. This type of subdivision is known as a binary tree. Each branch may contain many leaves, which are the smallest possible block. Each leave contains a maximum number of points.

The division of the original matrix into different blocks is done using a statistical method which uses geometric information about the model to separate the blocks. This analysis is called principal component analysis (PCA). This methodolody uses the covariance matrix of the position of the physical nodes. An eigenvalue extraction is performed and the eigenvalues and eigenvectors are used to separate the points using the principal

variability patterns present in the geometric model. Consider the covariance matrix of the geometric position of the physical nodes matrix P, shown in Eq. (13).

$$C = cov(P) \tag{13}$$

An eigenvalue extraction is performed in the matrix C,

$$Cv = \lambda v \tag{14}$$

where v is the eigenvector and λ is the eigenvalue of matrix C.

Let v_{max} be the eigenvector from v associated with the biggest eigenvalue λ_{max}. Then, v_{max} points to the direction of highest variability. The centroid of the model is calculated and the separation is performed according to Eq. (15).

$$p_i = (x_i - X_g)v_{max} \tag{15}$$

This separation technique allows the construction of the binary tree to a pre determined maximum number of points in each leave. This creates a separation plane which is orthogonal to the eigenvector of the covariance. This plane is situated in the centroid of the geometrical shape of the model. The definition of clusters 1 and 2 is shown in Eq. (16).

$$\begin{cases} if\ p_i > 0\ then\ x_i \in Cl_1 \\ if\ p_i < 0\ then\ x_i \in Cl_2 \end{cases} \tag{16}$$

This process will be applied recursevely to clusters 1 and 2 until both contain some (small and independent of N) n_{min} prescribed value or less number of points. Every cluster except the last one is named branch. The last one is called a leave.

3.2 Low-rank approximation

The rank of a matrix A is the number of linear independent lines or columns from A. This also describes the dimension of the vector space generated by its columns or rows. The rank of a matrix is one of its fundamental characteristics. It is possible to describe a matrix $A \in \mathbb{R}^{nxm}$ of rank k using the product of two vectors $U \in \mathbb{R}^{nxk}$ and $V \in \mathbb{R}^{kxm}$ as shown in Eq. (17).

$$A = UV \tag{17}$$

The representation (17) is called outer-product form.

The matrix-vector multiplication of the BEM can be written as shown in Eq. (18). The number of operations necessary for the multiplication Ax is $2mn$, while the multiplication $U(Vx)$ requires $2k(m + n - 1)$ operations.

$$Ax = U(Vx) \tag{18}$$

Matrices that have a rank that is relatively small compared with their dimension are one of the basic structures for the efficiency of hierarchical matrices (BEBENDORF, 2008). If matrix A is square, i.e. $A \in \mathbb{R}^{nxn}$, then the matrix-vector multiplication shown in Eq. (18) will need $4k(n-1/2)$ operations to be evaluated. Matrix A will be considered low rank when the condition in Eq. (19) is satisfied.

$$2kn \leq n^2 \tag{19}$$

Equation (19) shows the sufficient condition for which the representation shown in equation (17) is advantageous. When the rank of a matrix is considerably lower than their dimensions, then the matrix is considered to be a low rank matrix.

The BEM generates fully populated non-symmetric matrices. This results in high computational costs for large scale models, both in storage and computation. The influence matrices which results from integral equations have no explicit structure in general. This means that the matrices are not necessarily low rank matrices. It is possible, however, to approximate this matrix into a low rank matrix. This process is known as a low rank approximation. This approximation is defined as a minimization problem, in which the approximated matrix contains a lower rank than the original matrix.

Let $A \in \mathbb{R}^{nxn}$ be a given matrix having the entries obtained by BEM. Let $\tilde{A} \in \mathbb{R}^{nxn}$ be an approximation of A such that the error between A and $\tilde{A}$ is described by Eq. (20).

$$||A - \tilde{A}||_F \leq \epsilon ||A||_F \tag{20}$$

where $||*||_F$ is the Frobenius norm of variable $*$ and ϵ is the precision of the approximated matrix.

The approximated matrix $\tilde{A}$ can be obtained using several methods, including the adaptive cross approximation and the singular value decomposition.

Adaptive cross-approximation (ACA)

In this work, suitable matrices are approximated to low-rank matrices using the adptive cross-approximation (ACA). The approximation obtained is stored in the form shown in Eq. (21).

$$\tilde{A} = UV \tag{21}$$

The methodology for obtaining vectors U and V is the ACA, which adaptively determines the rank k of the approximation to a precision ϵ. For each step k, the error for the approximation $\tilde{A^k}$ is given by Eq. (22).

$$||A - \tilde{A^k}||_F \leq ||A - A^{\tilde{k-1}}||_F \approx ||\tilde{A^k} - A^{\tilde{k-1}}||_F = ||a_k b_k||_F \tag{22}$$

where a_k and b_k are the vectors created at step k of the approximation. This step determines the rank of the low-rank approximated matrix $\tilde{A^k}$. The relative error is given by Eq. (23).

$$\epsilon = \frac{||a_k b_k||}{\tilde{A^1}} \tag{23}$$

Using this methodology, matrix A doesn't need to be previously calculated in order to estimate the error. Only the lines and columns necessary for the approximation are calculated.

4 RESULTS

The results were obtained using the described methodology for different frequencies and the resonance modes were identified by identifying the absolute pressure peaks in the frequency response plot. The algorithms are implemented in Matlab 12.0 and all the results were obtained in a Sony laptop computer Intel Core i5 2.67 GHz with 4 GB of RAM memory and equiped with Windows 10 operational system.

Two configurations were assessed for the bidimensional acoustic room with holes. Both are square rooms, 1 meter by 1 meter. The holes are inserted as rigid walls and the sum of the cross section areas of the holes is equal to 12 % the total area of the square room. A physical interpretation of this problem is a cubic room crossed by paralell rigid tubes. Two models were created to determine the position of the center of the holes, and the results are divided into two categories: uniformly and randomly distributed holes.

Recent work has been made into scattering and inner room behaviour of these kind of acoustic holes or pipes. Small tube-like structures had been used by (CHEN, 2004) to study the effect of damping specific frequencies band and phononic crystals are suggested to control phonons, sound and other mechanical waves. Scattering from this type of circular acoustic objects are studied for high frequency waves using XIBEM using IGA in (PEAKE; TREVELYAN; COATES, 2015).

4.1 Square bidimensional room with uniformly distributed holes

The first model is a bidimensional square acoustic room with 25 equal radius r uniformly distributed holes. Each circular hole has radius r. The sum of the areas of the circular holes corresponds to 12.47 % of the total area of the room. The boundary conditions for the room are shown in Figure (1).

This model consists of 120 constant elements for the enclosure and 200 constant elements for the 25 holes.

This model was solved using both the traditional BEM and the ACA BEM, obtaining similar results. The resonance frequencies obtained were equal for both methods, but a

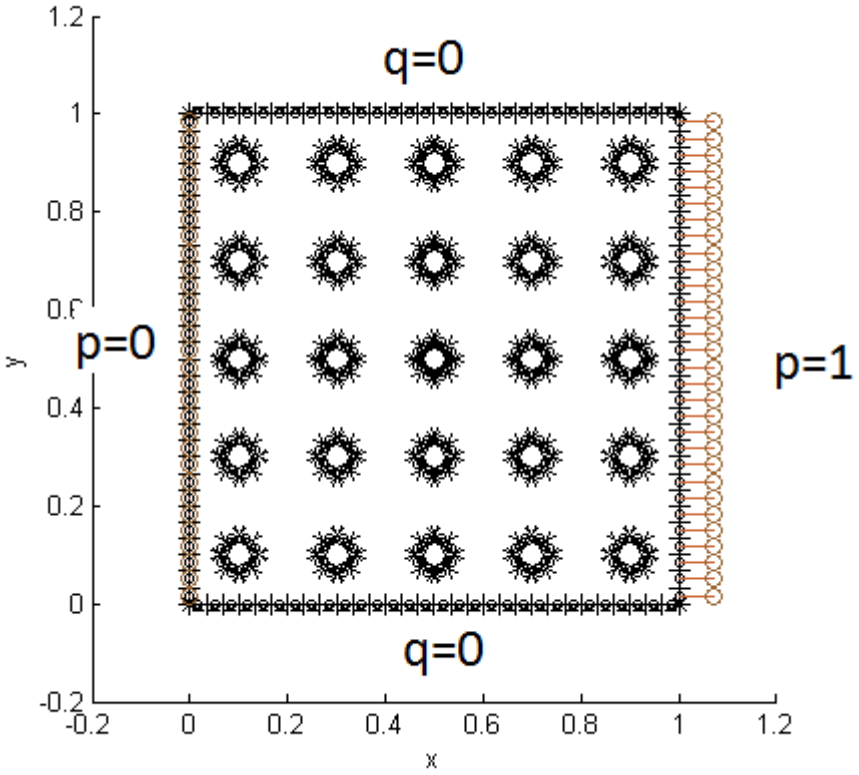

Figure 1: Boundary conditions for the square room with uniformly distributed holes. q is shorthand for pressure flux and p is shorthand for acoustic pressure [atm].

Mode frequency [rad/s]	1	2	3	4
BEM	2126.81	2892.46	4168.55	5189.42
ACA BEM	2126.81	2892.46	4168.55	5189.42

Table 1: Resonance mode frequencies for both BEM and ACA BEM models.

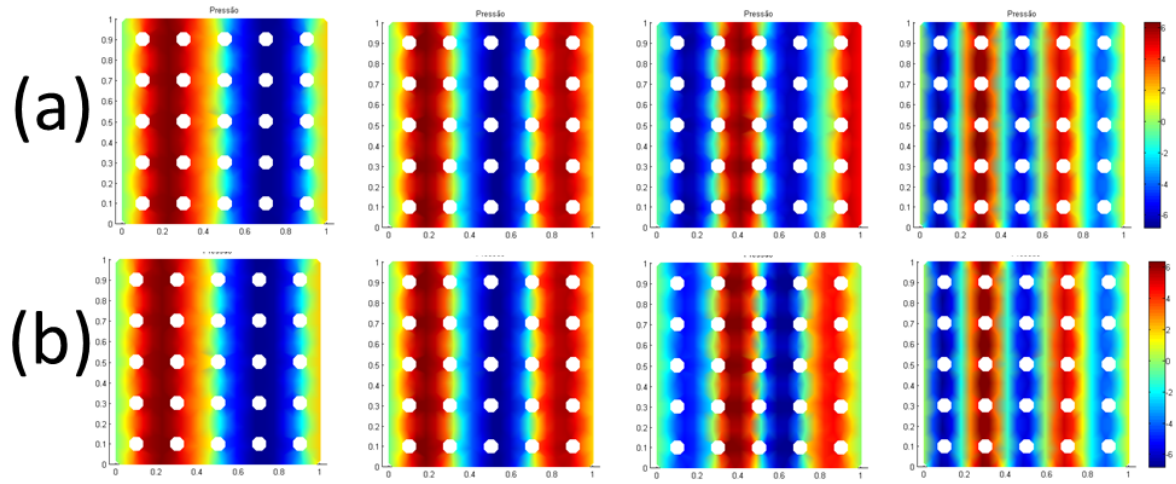

Figure 2: Mode comparison between BEM and ACA BEM results for uniformly distributed holes for the 4 first resonance frequencies. (a) BEM results. (b) ACA BEM results.

widening of the frequency resolution could produce a better estimation. Table (1) shows the frequency obtained for the three first resonance modes identified.

The ACA methodology was able to reduce the total processing time from 96 to 62 minutes (35% improvement) from the approximation of 159700 values. The use of the GMRES resulted in a major bottleneck for speed in the computation. The time needed to obtain the ACA BEM results using GMRES was nearly 3 times the processing time using a direct solver from Matlab 12.0 ($\approx$ 10000 vs $\approx$ 3000 seconds). This is expected for problems with less than $\approx$ 100000 elements, where the GMRES becomes more competitive. The resonance modes obtained for both methods are equivalent and can be seen in Figure (2).

4.2 Square bidimensional room with randomly distributed holes

This next example consists of the same acoustic enclosure filled with 20 randomly distributed holes. The radius of the holes is constant and the sum of the areas of the holes is equal to 12.47 % of the total area of the enclosure. To distribute the position of the center of the circular holes, two limitations were imposed: the centers must be at least αr from the walls and $2\alpha r$ from each other. The constant $\alpha > 1$ guarantees that the holes don't touch either each other nor the walls of the room. For this step of the assessment, three models were created in order to study the convergence of the model. The boundary conditions are shown in Figure (3).

Table (2) shows the different models used in this work. Three different resolution meshes are created: coarse, regular and refined. This convergence was performed for both the BEM and ACA BEM models.

The resonance frequencies obtained are the same as the ones shown in Table (1). The mode frequencies differ a lot from the model with uniformly positioned holes, but the BEM and the ACA BEM obtained virtually the same results for every mode. This is shown in Figure (4).

It is possible to discern the holes interference in the mode shapes shown in Figure

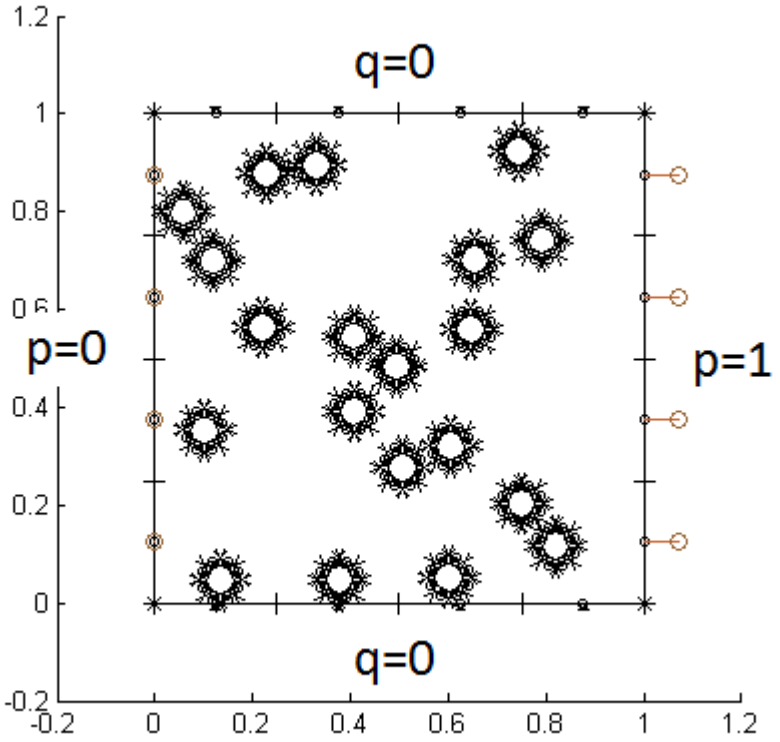

Figure 3: Boundary conditions for the square room with randomly distributed holes. q is shorthand for pressure flux and p is shorthand for acoustic pressure [atm].

Model	Elements for the enclosure	Elements for the holes
Coarse mesh	16	160
Regular mesh	32	160
Refined mesh	64	160

Table 2: Number of constant elements used for the different models.

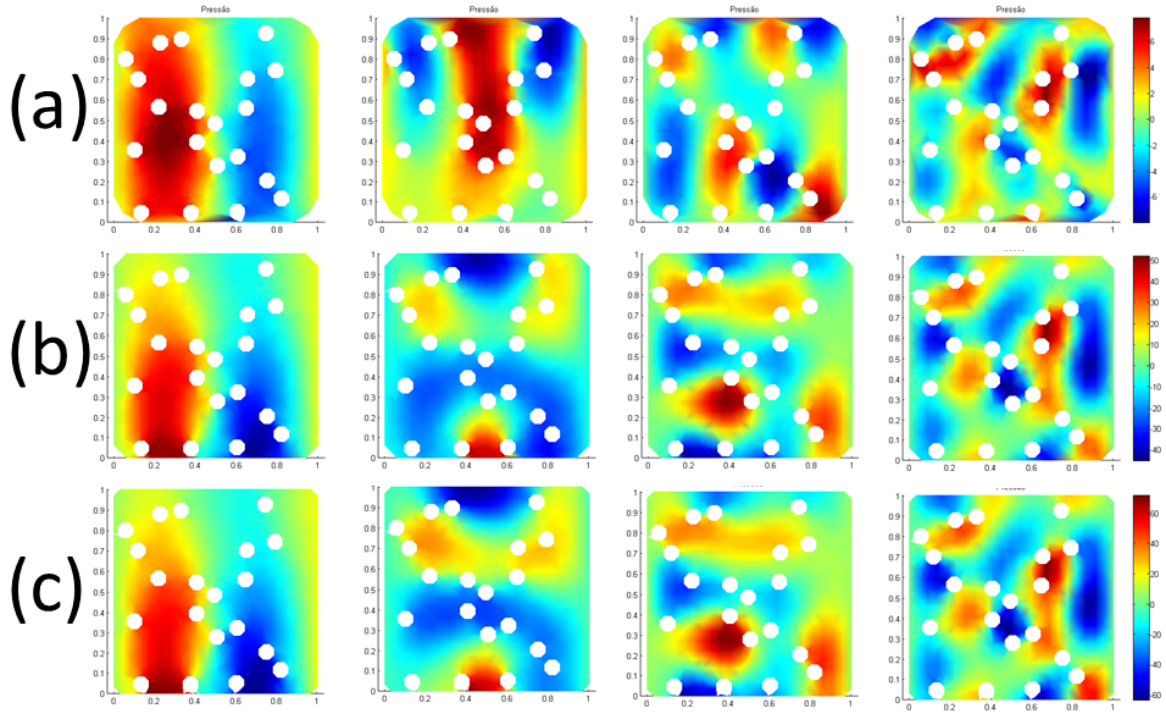

Figure 4: Mode comparison between BEM and ACA BEM results for randomly distributed holes for the 4 first resonance frequencies. (a) Coarse mesh. (b) Regular mesh. (c) Refined mesh.

(4). The mode shapes obtained by the BEM and the ACA BEM are very similar and, in this case, the ACA BEM took more time to compute than the BEM model, but this effect can be caused by the different programs running during the simulation. This effect can be reduced if the processing from Matlab 12.0 is controlled.

5 CONCLUSION

This paper presented the methodology for solving a fast boundary element method (ACA BEM) bidimensional model consisting of a square enclosure filled with both uniformly and randomly positioned holes. The results showed that both the BEM and the ACA BEM agreed upon the general acoustic behaviour of the models. The ACA procedure was able to speed up the processing of the BEM without loosing any apparent accuracy. For this problem, the GMRES was shown to not be suitable to speed the solution of the problem, and a direct solver was much faster, mainly due to the efficiency loss from Matlab's high level functions such as "for's" and data storing limitations.

The perspectives for this work is to approach the scattering problem of circular tubes and other geometry sccatterers. Apply the ACA BEM to tridimensional acoustic problems and changing the geometry of the tubes, concentrating the tubes in different areas of the enclosure and controlling the processing from the PC during simulation to obtain more accurate measurements on the efficiency of each method.

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