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REDUCING ORDER METHODS APPLIED TO RESERVOIR SIMULATION

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Abstract. *Results obtained by numerical simulations techniques are used in the whole productive life of the reservoir, since exploration until enhanced oil recovery. Numerical simulations involves many cells and heterogeneities and are still limited by the computational time and memory. Reducing Order Methods are a solution to these problems. These methods allow the transformation of high-dimensional models into meaningful representations. It reduces the dimension of the matrices used during the simulations, and consequently, the time and effort. One of the methods used to get a reduced model is the Proper Orthogonal Decomposition (POD). In this work, the mathematical model and equations of a considered reservoir are first presented, and in sequence the discrete system obtained by Finite Difference and Finite Volumes methods. Then, the POD procedure will be described and applied to the problem considered. Finally, the size of new matrices and pressures will be evaluated before and after the reduction, as well as the error involved. The results obtained after the reduction agreed with the physical of the problem and, as expected, the number of unknowns reduced significantly.*

Keywords: *Reservoir Simulation, Reduced Order Methods, Proper Orthogonal Decomposition*

1 INTRODUCTION

The flow in the porous media is controlled by the Hydraulic Diffusivity Equation (HDE), which governs the pressure behavior in time and space. This equation can be solved by numerical methods, for example finite difference, finite volumes and finite elements method. The application of numerical methods reduces the problem to a system of equations that, when solved, results in the pressure and saturations (in case of multiphasic flow) to each point of the discretized model.

Despite of the computational memory capacity available nowadays, reservoir discretization implicates in a huge number of cells and heterogeneities. Thus, reservoir simulations are still limited by the computational capacity available. The modelling become complex as the numerical model of the reservoir approaches to the real one, which involves properties changes in the centimeter scale. Moreover, depending on the method, the system to be solved is non-linear, involving more computational effort during the solution.

The application of Reducing Order Methods (ROM) is a way to reduce the order of the problems that will be solved, what means the reduction of unknowns of the system. Applying this method, a reduced model will be generated, and must be able to capture the essential dynamical behavior and also preserve critical properties of the large models (Sava, 2012). The ROM's have been widely used in many areas of engineering in the last years, such as: chemical reactors, aerodynamics and automation. In Petroleum Reservoir Engineering, this technique application is recent.

Applying ROM in the scope of petroleum reservoirs, the full model is simulated just once. The reduced model is then obtained after some algebraic operations based on the results from the full model simulation. Next simulations, that can be many, will apply the reduced model obtained, reducing the size of the system solved. Therefore, the knowledge and control of the computational code of the simulator is essential, either for the reduced model or for the full one.

Among the existing methods used to get a reduced model, the Proper Orthogonal Decomposition (POD) is the one commonly used in non-linear problems (CARDOSO, 2009). The main idea of POD is the high dimensional system projection in a reduced dimension subspace, that contains the essential dynamics of the system (Sava,2012). This projection is done based on matrices obtained by the data from the full model simulation. Projecting the system, the number of unknowns is reduced, what also effects the computational time and effort.

In this work, the POD will be applied to oil flow in rectangular and heterogeneous reservoirs. The equations that control the pressure in the porous media are presented in the next section. In sequence, the problem is discretized and Finite Difference and Finite Volume methods are applied, reducing the problem to a system of linear equations. Finally, the POD procedure is presented and used in a problem example.

2 RESERVOIR PROBLEM FORMULATION

Reservoir simulation determines the saturation, pressure and temperature for the domain in each time step. In this work, the flow is considered isothermal and monophasic. The pressure is controlled by the Hydraulic Diffusivity Equation (HDE). This equation is obtained from the association of three basic equations: the continuity equation, the Darcy's law and a state equation (Rosa, 2011).

Considering an heterogeneous reservoir without gravitational effects and a fluid with constant properties flowing trough it, the HDE is as Eq. (1) and given by

$$\phi C_t \frac{\partial p}{\partial t} - \vec{\nabla} \cdot \left(\frac{k}{\mu} \vec{\nabla} p \right) = \tilde{q}, \tag{1}$$

where ϕ is the media porosity, k the permeability, μ the viscosity and C_t the total compressibility. The first term of Eq. 1) is called accumulation term, and represents the accumulation of a mass of fluid per unit of time and volume. The second term, the transmissibility, represents the amount of fluid that flows in or out a volume unit due to the pressure differential. The $\tilde{q}$ term represents the external mass fonts acting in the reservoir, for example wells.

The HDE in the form presented previously, Eq. (1), is a partial differential equation. In this way, its necessary to define boundary and initial conditions to ensure that the solution will exist and be unique. These conditions, initial and boundary, will be discussed in the next subsections.

2.1 Initial Condition

Usually we adopt as an initial condition the equilibrium of the fluids in the reservoir, what means that the pressure is constant in all its extension. Then, the initial condition of a reservoir, for $t = 0$, is as:

$$p(x, y, t = 0) = \bar{p}. \tag{2}$$

where $\bar{p}$ is a prescribed pressure.

2.2 Boundary Conditions

The boundary conditions are necessary to solve partial differential equations since it is essential to determine the bounds of the domain that will be studied, and also insert a disequilibrium in the system. In the way of reservoir simulation, there are two kinds of boundary conditions: internal boundary condition and external boundary condition.

The external boundary conditions are related to the conditions prescribed in the limits of the reservoir. This condition can be a prescribed pressure $\bar{p}$, as Eq. (3):

$$p(x, y, t) = \bar{p} \quad (x, y) \quad \epsilon \quad \Gamma, \tag{3}$$

or also a prescribed flow, as Eq. (4):

$$\frac{\partial p}{\partial \bar{n}} = q(x, y, t) = \bar{q} \quad (x, y) \quad \epsilon \quad \Gamma, \tag{4}$$

where n is the direction of the flow q and Γ the domain that it is prescribed.

If there are producers or injectors wells in the reservoir, there will be internal conditions. These wells can present the bottom hole pressure or mass flow constant, in a position of coordinates x_w, y_w, according to Eq. (5) and Eq. (6):

$$p(x_w, y_w, t) = p_w, \tag{5}$$

$$\frac{\partial p}{\partial \bar{n}} = q(x_w, y_w, t) = q_w. \tag{6}$$

2.3 Adimensionalization

The adimensionalization of the differential equation that govern the problem of the fluid flow in reservoir, as well as the numerical solution obtained, makes the solution and the interpretation of the results easier. It also does not include conversion of units and the matrices used do not have problems with conditioning. Some mathematical manipulations are needed, and it will be explained in sequence.

We define the adimensional pressure as Eq. (7),

$$p_D(x_D, t_D) = [p(x,t) - p_o]\frac{k2\pi h}{\mu q_w}, \tag{7}$$

where p_o is a known pressure, took as a base pressure.

The adimensional time, as Eq(8),

$$t_D = \frac{\eta t}{L^2}, \tag{8}$$

where L is a known length and η the hydraulic diffusivity constant. For the spatial domain, in direction x, the adimensionalized variable x_D is defined as Eq. (9)

$$x_D = \frac{x}{L}, \tag{9}$$

what is analogue for the direction y.

Replacing the variables defined before, and considering squared blocks, we get that Eq. (1) becomes Eq. (10):

$$\frac{\Delta x_D^2}{2\pi}\frac{\partial p_D}{\partial t_D} = \frac{\Delta x_D^2}{2\pi}\vec{\nabla}_D \cdot \left(k_D \vec{\nabla} p_D\right) + \tilde{q_D}. \tag{10}$$

In the next sections, numerical methods will be applied to Eq. 10, in order to obtain the solution, pressure, in each point of the domain.

3 DISCRETE FORMULATION

The Hydraulic Diffusivity Equation presented in Eq. (10) does not present an analytical solution, except for the simplified case one-dimensional. On this way, it is necessary to apply some numerical methods to solve the equation. According to Maliska (2004), the task of a numerical method is solve one or more differential equations replacing the derivatives with algebraic expressions involving the unknowns. Applying the numerical method, the solution, pressure, will be obtained in discrete points of the domain, in finite time steps.

To obtain the pressure behaviour in time, the spatial part of Eq. (10) will be solved by the Finite Volume Method and the temporal by the Finite Difference Method, as discussed in the next sections.

3.1 Finite Volume Method

The spatial domain is first subdivided in square blocks, each one with constant permeability. Analysing a block that is not on the boundary of the reservoir, the block i, j in Fig. (1), it's possible to notice that there are four flows in this block: q_{DW} on the west direction, q_{DE} in the east, q_{DN} to north and q_{DS} to south, how is showed in Fig. (2).

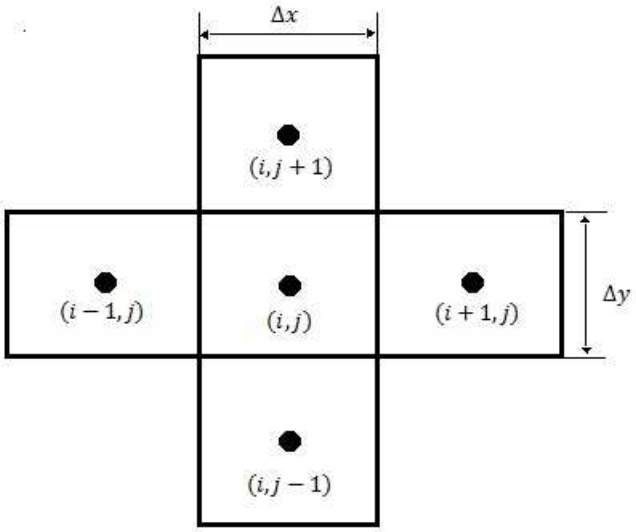

Figure 1: Discretizated central block

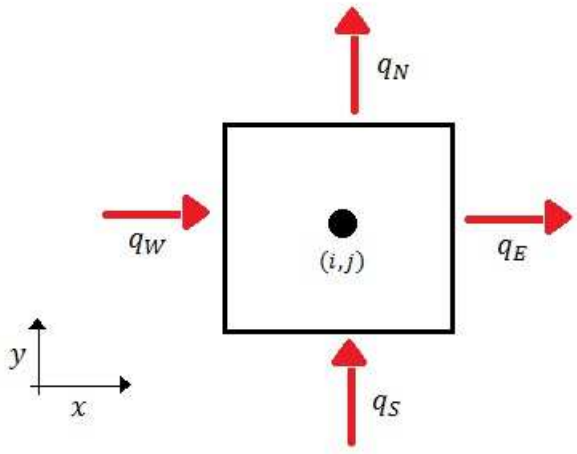

Figure 2: Flow in a centered block

Comparing the flow term in Eq. (10) and Darcy's equation, we conclude that

$$k_D \vec{\nabla} p_D = -q_D. \tag{11}$$

Applying Eq. (11) to the flow term in Eq. (10),

$$\vec{\nabla}_D(k_D\vec{\nabla}p_D) = -\vec{\nabla}\cdot q_D. \tag{12}$$

Considering that there is no flow in the diagonal direction, we apply Eq. (10) in Eq. (12), with the four flows presented before. Integrating the resulting expression in the volume of a central block, we obtain that

$$\int_V \frac{\partial p_D}{\partial t_D}dx_D dy_D = -\int_V \vec{\nabla}\cdot q_D dx_D dy_D + \int_V \frac{2\pi}{\Delta x_D \Delta y_D}\tilde{q_D}dx_D dy_D. \tag{13}$$

Applying the Divergence Theorem in Eq. (13),

$$\frac{\partial}{\partial t_D}\int_V p_D dx_D dy_D = -\int_S \hat{n}\cdot\vec{q_D}d\ell + \frac{2\pi\Delta x_D\Delta y_D}{\Delta x_D\Delta y_D}\tilde{q_D}, \tag{14}$$

where $\hat{n}$ is the vector normal to the surface ℓ.

Analysing the flows explained before to Eq. (14), its possible to obtain,

$$\frac{\partial p_D}{\partial t_D}\Delta x_D = -(q_{DN} + q_{DE} - q_{DW} - q_{DS}) + \frac{2\pi}{\Delta x_D\Delta y_D}\tilde{q_D}. \tag{15}$$

The flows of Eq. (15) need to be writed as a function of pressure, the unknown of the problem. The fluid flows from a block with a permeability A to other, with a permeability B, and the flux is calculated based on its equivalent permeability. Considering Darcy's Law, the equivalent permeability in the four directions is calculated by the harmonic mean of the blocks where the flow happens. Making it to each direction, we obtain that Eq. (15) yields to

$$\frac{\partial p_D}{\partial t_D}\Delta x_D = -[k_{DPW}(p_{D_{i,j}} - p_{D_{i-1,j}}) + k_{DPS}(p_{D_{i,j}} - p_{D_{i,j-1}}) - \dots$$

$$k_{DPE}(p_{D_{i+1,j}} - p_{D_{i-1,j}}) - k_{DPN}(p_{D_{i,j+1}} - p_{D_{i,j}})] + \frac{2\pi}{\Delta x_D\Delta y_D}\tilde{q_D}. \tag{16}$$

In Eq. (16), each permeability corresponds to the adimensional equivalent, from the point analized, into a direction (north, south, east, west).

Next section, the part of Eq. (10) related to time will be discretized by finite difference method.

3.2 Finite Difference Method

Related to time, the finite difference method can be applied by two ways: implicit or explicit. The explicit is computationally cheaper, but requires some criteria guaranteed to be stable. These criteria are directly related to the step time. The implicit method, naturally stable, will be used in this work. It means that the pressures in Eq. (16) will be analysed in the time step $n+1$. Than, we can write the time portion of Eq. (16) as

$$\frac{\partial p_D}{\partial t_D}\Delta x_D = \frac{p^{n+1}_{D_{i,j}} - p^{n}_{D_{i,j}}}{\Delta t_D}. \tag{17}$$

We finally get

$$\frac{p_{D_{i,j}}^{n+1} - p_{D_{i,j}}^{n}}{\Delta t_D} = -[k_{DPW}(p_{D_{i,j}}^{n+1} - p_{D_{i-1,j}}^{n+1}) + k_{DPS}(p_{D_{i,j}}^{n+1} - p_{D_{i,j-1}}^{n+1}) - \dots$$

$$k_{DPE}(p_{D_{i+1,j}}^{n+1} - p_{D_{i-1,j}}^{n+1}) - k_{DPN}(p_{D_{i,j+1}}^{n+1} - p_{D_{i,j}}^{n+1})] + \frac{2\pi}{\Delta x_D \Delta y_D}\tilde{q_D}. \tag{18}$$

The Eq. (18) can be written as a linear system, and then be solved in all the time steps, finding the pressure in each block of the reservoir.

We define the transmissibility matrix as the one that has the properties of the porous media, and is the one that multiplies the pressure vector p_D. T is a pentadiagonal matrix, since the problem solved in this work is two dimensional. The diagonal matrix that multiplies the time variation is called D, and the vector $\bar{f}$ is the source term, that contains the terms related to the wells acting in the reservoir. So, it's possible to write Eq. (18) as

$$D(p_D^{n+1} - p_D^n) = Tp_D^{n+1} + \bar{f}. \tag{19}$$

The system described in Eq. (19) is solved in each time step, and has the size of the blocks in the reservoir.

4 PROPER ORTHOGONAL DECOMPOSITION

The POD method is classified as a projection based method (Sava, 2012). The full model is projected in a low dimensional subspace, that has all the important characteristics and dynamics of the system. In order to generate a basis to reduce the system, the simulation of the full model is first done. During the solution of the problem described before, each time step, or snapshot, generates a array of pressures, x^i, with length n, corresponding the pressures in the blocks of the mesh or points of the discrete domain. Organizing all s snapshots pressure arrays in a matrix, we obtain

$$X = x^1, x^2, ..., x^s, \tag{20}$$

and X is a matrix of n lines and s columns.

The POD procedure is well established in many publications. Here, we will follow the procedure of Cardoso (2009). First, the mean of pressures during the snapshots, $\bar{x}$, is computed

$$\bar{x} = \frac{1}{S}\sum_{i=1}^{S} x^i, \tag{21}$$

then the data matrix $\hat{X}$ is obtained by

$$\hat{X} = [x^1 - \bar{x}, x^2 - \bar{x}, ..., x^s - \bar{x}]. \tag{22}$$

Numerically, $\hat{X}$ represents the variation of the pressure in time better than X. It occurs because $\hat{X}$ contains the time variation of the pressure in each block, instead of the pressure, like in X. Since the main objective of the ROM is to disconsider the effect of blocks that do not

variate with time, the operation made to obtain $\hat{X}$ is helpful. Moreover, it helps in the cut of unknowns that will be done during the ROM procedure.

The aim of the POD is to find an orthogonal basis such that the total mean squared error between the snapshots and their projections on the reduced subspace is minimized (Sava,2012). This minimization problem is reduced to an eigenvalue and eigenvector problem related to the covariance matrix, C:

$$C = X^T X. \tag{23}$$

To find the projection basis Φ, the eigenvalue problem is solved

$$C\Psi = \lambda\Psi, \tag{24}$$

where Ψ represents the eigenvectors and λ the eigenvalues of C.

The POD basis functions, Φ are given by a linear combinations of the snapshots,

$$\Phi = \hat{X}\Psi. \tag{25}$$

The Φ matrix is the orthogonal matrix projection, that results in a linear transformation from the full model system into reduced one. In order to reduce the order of the projected (or reduced) subspace, only the higher eigenvectors of Ψ are selected to build the Φ basis. Those higher eigenvalues are the correspondents of the higher eigenvalues of the problem.

The next step is to select the higher eigenvalues, and the correspondent eigenvectors. In this way, we can apply the term energy, E_l, defined in Eq. (26)

$$E_l = \frac{\sum_{i=1}^{l} \lambda_i}{E_t}. \tag{26}$$

In Eq. (26), l corresponds to the number of snapshots selected (the higher ones), and E_t corresponds to the sum of all the eigenvalues. The near E_l is to 1, the better is the cut, and better will be the reduced representation.

The basis matrix Φ_l corresponds to the eigenvectors selected by the higher l eigenvalues, and it will have only l columns, instead of s. The reduced state vector z, with a length l, can be calculated by Eq. (27)

$$x \simeq \phi_l z + \bar{x}. \tag{27}$$

Defining as N_b the number of blocks in the reservoir, we get that $l << N_b$, what significantly reduces the computational time and effort.

Replacing x defined in Eq. (27) in p_D of Eq. (19), we obtain that

$$D(\Phi_l z_D^{n+1} - \phi_l z_D^n) = T(\Phi_l z_D^{n+1} + \bar{x}) + \bar{f}. \tag{28}$$

Pre multiplying Eq. (28) by Φ_l' we obtain,

$$\Phi_l' D(\Phi_l z_D^{n+1} - \phi_l z_D^n) = \Phi_l' T(\Phi_l z_D^{n+1} + \bar{x}) + \Phi_l' \bar{f}. \tag{29}$$

In the system of Eq. (29), the new unknown is z, that presents a length l. It means that, the new system to be calculated has l equations and l unknowns, instead of N_b, reducing the computational time and effort, main objective of ROM.

5 NUMERICAL EXAMPLE

The problem simulated is a squared reservoir with no-flow boundaries. There are 20 blocks in x direction and 20 in the y one. Initially, the reservoir is in a constant pressure $p_i = 0$, and a well with a constant flow rate is located in the middle of the reservoir. The geometry of the reservoir is shown in Fig. (3).

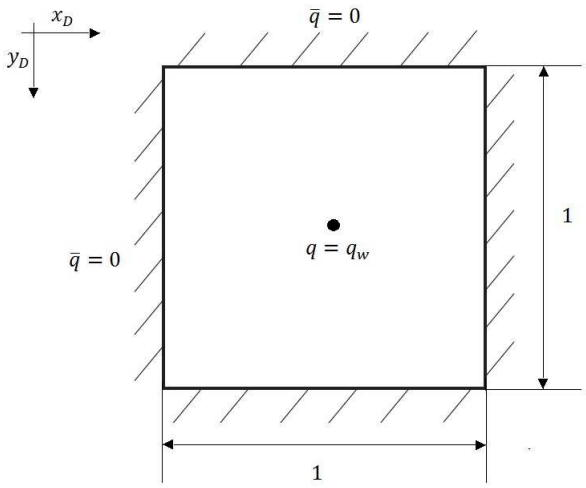

Figure 3: Geometry and boundary conditions of the problem simulated

The Fig.(4) shows the permeability matrix obtained randomly.

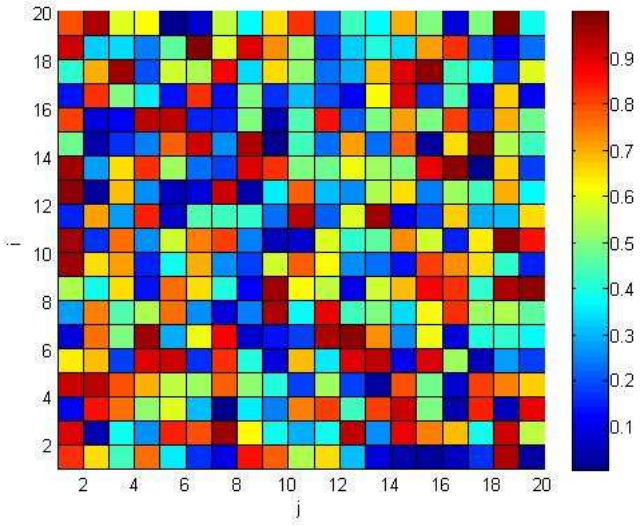

Figure 4: Permeability field of the simulated reservoir

The final adimensional time is $t_D = 10$, and 10 time steps were used to subdivide the time in equal intervals. The well is opened when the simulation starts, and its bottom pressure changes in time, to maintain the production rate constant.

5.1 Results

Simulating the flow of oil in the well, the pressure in the line 10, where the well is placed, is plotted for some time steps in Fig.(5).

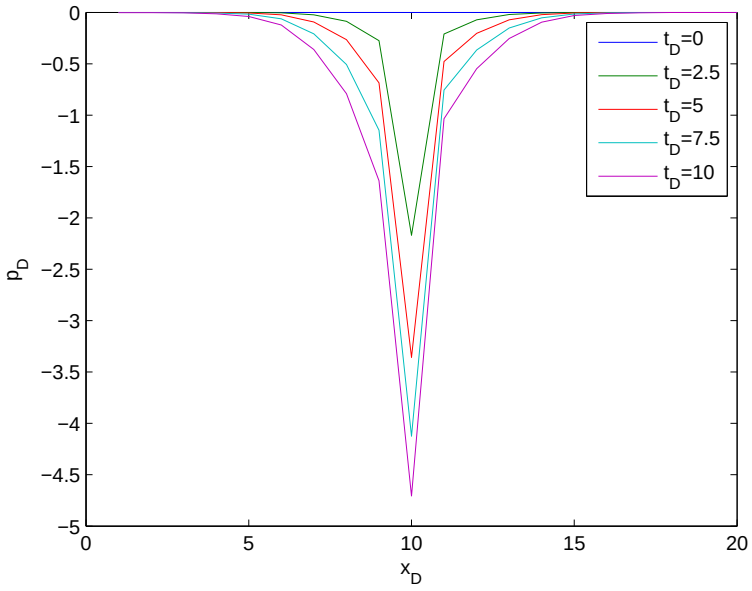

Figure 5: Time response of the pressure in the line of the well

For this example, there are 400 unknowns (number of blocks) to be calculated in each time step. The matrix of data, X, used to compute the basis of the reduced order, has dimensions 400 lines and 10 columns, and the software took a time of 0.203 seconds to simulate the 10 time steps. It's a smart simulation time, because of the simplicity of the problem and its symmetry, since just the permeability varies spatially.

With the results of the full model reservoir, the application of Proper Orthogonal Decomposition is possible. A new program routine is elaborated, using the X matrix obtained before. The eigenvalues and eigenvectors are calculated by a Single Value Decomposition, and the POD basis is obtained. The 10 eigenvalues obtained varies a lot, as can be seen in Fig.(6).

The eigenvalues behaviour observed is related to the symmetry of the reservoir. Moreover, some blocks have almost the same pressure varying with time, because the pressure wave caused by the well do not reaches the boundaries of the reservoir. This behaviour also means that just few eigenvalues represent the process, and the energy of the cut is almost equal to 1.

The first five eigenvalues are the biggest ones, so the eigenvectors associated were selected to construct the POD basis. The energy of the cutting was calculated by Eq. (26), and it differs from one in the nine decimal case.

Using the POD basis constructed, the problem is simulated again, using Eq. (29). The new simulation time is 0.05 seconds, and instead of 400 values of pressure calculated in each time step, 5 blocks are considered. Comparing the last time step of the full model with the reduced model, we get that the pressures follow the same pattern, just with a few difference between then, as can be noted in Fig.(7).

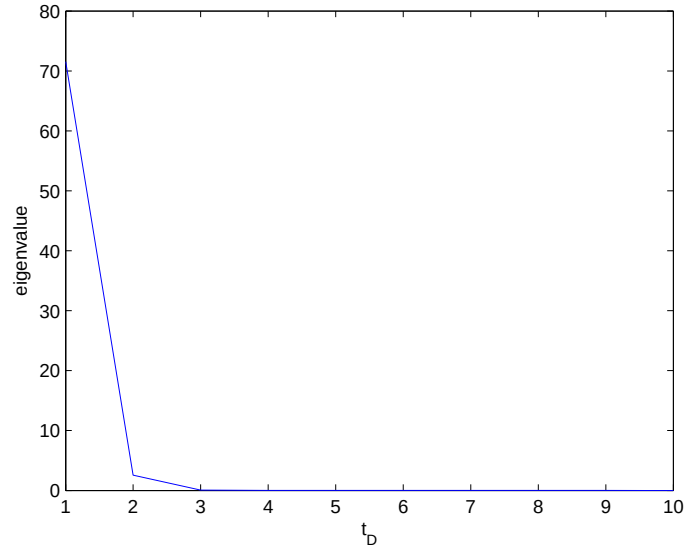

Figure 6: Eigenvalues of the simulated reservoir

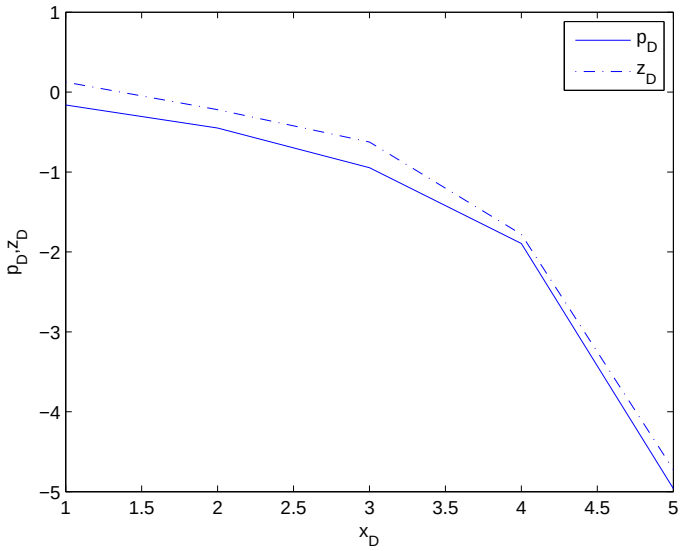

Figure 7: Full model and reduced model pressure to $t_D = 10$

As expected, the reduced model considered the symmetry of the reservoir. As compared in Fig. (7), the reduced state vector has the same behaviour of the first five nodes on the right side of the reservoir. Being a square reservoir, the pressure in the left side will change almost equal to the left side, what also occurs in up and down directions. Noticing that the right side presents higher permeabilities, and consequently higher flux velocities, the pressure changes more than in the right side. The pressure in the last five nodes in both of the sides does not change so much with time, as showed in Fig. (4). In this way, only the first five nodes varies the pressure sufficiently to be considered in the reduced model.

The error between the pressure of the full model and the reduced one was calculated in the last time step, as presented in Table (1). The values present a difference relatively high,

but it decreases as the node becomes far from the well. This difference is also related to the adimensional value, and it tends to be smaller when the pressure is recalculated by Eq. (7) with an unity.

Table 1: Percentage error: reduced-state vector and pressure.

Node	$\|p_D\|$	$\|z\_D\|$	Error	Percentage error
1	0,197	0,111	0,086	43,47%
2	0,376	0,182	0,194	51,56%
3	0,801	0,494	0,307	38,33%
4	1,840	1,348	0,491	26,71%
5	4,169	3,483	0,685	16,44%

6 CONCLUSION

Reservoir Simulation is an important area of Petroleum Engineering, and the results of this technique are related to how much the model simulated looks like the real one. These models have many cells and heterogeneities, what makes the simulation takes a long time and high computational effort. The Reduced Order Methods are a new important simulation skill that reduces the simulation time, but is still being studied and tried in reservoir problems.

The Reduced Order Methods can be a solution for the time and effort spent in reservoir simulation. It's a technique that provides a good representation of the full model, just implementing a simple routine with the simulator and evaluating some characteristics of the results from the full model.

The results obtained in this work are as expected, respecting the problem symmetry and behaviour in time. The error obtained is relatively high, but decreases as the node analysed becomes far from the well. It happens because of the high differential pressure in the block of the well, the fluid sink. Beyond that, the error is also high because of the adimensionalization, and will be smaller when it is undone.

For next works, new boundary conditions can be tested and the reduced model analysed in all the time steps. Moreover, new methods can be implemented, for example the TPWL (*Trajectory Piecewise Linearisation*), comparing the results and finding the better method to the case simulated.

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