A ROBUST CONDENSATION STRATEGY FOR STOCHASTIC DYNAMIC SYSTEMS

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Abstract. *In traditional design of engineering systems, it is normally assumed the mean values of the physical and mechanical properties. However, in real-world applications it may not characterize with reasonable accuracy the modifications on the dynamic behavior of the resulting systems induced by small changes on their design variables. Thus, it is interesting to perform a stochastic modeling strategy in order to take into account the presence of uncertainties. However, the stochastic finite element modeling of a more complex engineering structure composed by a large number of degrees of freedom, or its use in dynamic analyses requiring several evaluations such as in optimization and model updating, the computational cost can be prohibited or sometimes unfeasible. In these situations, the proposition of condensation strategy especially adapted for the resulting stochastic systems is interesting. This paper is devoted to the investigation of a robust model condensation strategy to reduce the random matrices of the stochastic system. The basis to be used is formed by a nominal basis evaluated by performing firstly an eigenvalue problem of the mean model enriched by static residues due to the small modifications introduced. To illustrate the main features and capabilities of the proposed strategy, numerical simulations were performed for a plate model in which the stochastic mass and stiffness matrices were generated by applying the so-called Karhunen-Loève expansion. The stochastic results are presented in terms frequency response function envelopes for the full and reduced stochastic dynamic systems subjected to a deterministic excitation.*

Keywords: *Parametric uncertainties, robust condensation, stochastic finite elements method, dynamics.*

1   INTRODUCTION

Engineering structures require reliability, durability and security allied with low cost. Deterministic finite elements method, also referred as Classic FEM, considers mean parameter values in order to characterize them. Mechanical and geometrical properties such as thickness, Young's module and mass density are estimated via metrological procedures and posteriorly receive statistical treatments that are always subjected to measurement errors and uncertainties. This along with simplifications applied to computational models, may lead the deterministic approach not to characterize the dynamic behavior with reasonable accuracy. Therefore, nowadays the stochastic finite elements method (SFEM) is becoming widely used in real-world engineering structures to take into account the presence of uncertainties and generate responses closer to reality, although requiring a more onerous formulation and higher computational efforts.

Stochastic simulations require multiple evaluations of the entire model. Time dispended in this process may turn prohibitive to compute exact evaluations performed over full matrices during each sample step. There are several methods capable of reducing dimensions of a structure (Craig, 2006). The main objective of this work is to propose a robust condensation strategy based in model updating techniques instead of extracting eigensolutions for each sample, as used in standard component-mode synthesis.

After a theoretical background concerning stochastic finite elements method and model reduction techniques, a numerical application for an aluminum thin plate was developed. Effectiveness of the robust condensation strategy proposed herein is then evidenced.

2   STOCHASTIC FINITE ELEMENT FORMULATION

In general, uncertainties are included in engineering models following non-parametric (Soize, 2000; Ritto et al., 2008) and parametric (de Lima et al., 2010a) approaches. The first method consists in introducing uncertain variables directly in global matrices, while the second one includes system perturbations by the addition of a stochastic part to deterministic matrices. SFEM, which permits a combination of classic and statistical analysis, is included in the second kind (Ghanem and Spanos, 1991; Schueller, 2001).

This section is devoted to formulate SFEM for an aluminum thin plate. Its dimensions are $330\,mm \times 390\,mm$, discretized by 10×10 plane rectangular finite elements mesh with four nodes and five degrees of freedom (DOFs) per node, which are three in-plane displacements u_0, v_0, w_0 and two cross-sections rotations θ_x, θ_y, comprising a total number of 605 DOFs. As boundary conditions, it will be considered a cantilever plate, clamped along y axis. Mechanical properties of the plate are presented in Table 1.

Table 1. Mean mechanical properties of the aluminum plate

Thickness (h) [mm]	Young's Module (E) [GPa]	Poisson ratio (v)	Mass density (ρ) [kg/m³]
2.5	70	0.34	2700

To include uncertain variables in the model, a parametrization process is required. Design parameters that may be subjected to fluctuations and random effects are selected and posteriorly factored out of mass and stiffness matrices. An immediate consequence of this process is the separation into membrane and bending effects (Rosa and de Lima, 2016), as shown in Eqs. (1).

$$\mathbf{K} = Eh\mathbf{K_m} + Eh^3\mathbf{K_b} \tag{1.a}$$

$$\mathbf{M} = \rho h\mathbf{M_m} + \rho h^3\mathbf{M_b} \tag{1.b}$$

where sub index m and b refer to membrane and bending effects, respectively.

A continuous random field $H(x,y,\theta)$ is a set of random variables (noted θ) indexed by physical coordinates x,y (for bi-dimensional structure such as a plate). Karhunem-Loève (KL) decomposition (or expansion) is a continuous representation for random fields expressed as the superposition of orthogonal random variables weighted by deterministic spatial functions (de Lima et al., 2010a). The discretized field $\hat{H}(x,y,\theta)$ obtained by KL decomposition and truncated in $r = n$ terms is presented in Eq. (2).

$$H(x,y,\theta) \cong \hat{H}(x,y,\theta) = \mu + \sum_{r=1}^{n} \sqrt{\lambda_r} f_r(x,y)\xi_r(\theta) \tag{2}$$

where $\mu = E\left[H(x,y,\theta)\right]$ is the expectation operator applied to the random field, $\xi_r(\theta)$ is the aleatory variables function and λ_r, $f_r(x,y)$ are the eigenvectors and eigenfunctions of the random field's bidimensional covariance function.

Ghanem and Spanos (1991) proposed a procedure to estimate λ_r and $f_r(x,y)$. For this bidimensional application, they suggest choosing an exponential covariance function. A function of this kind presents the separability property, allowing to decouple the problem into two unidimensional fields. Thus, the eigenfunction and eigenvalue solution become $f_r(x,y) = f_i(x)f_j(y)$ and $\lambda_r = \lambda_i\lambda_j$, respectively. The full procedure is described in details and was successfully applied in recent works of Ribeiro (2015), Rosa and de Lima (2016).

Once eigenfunctions and eigenvalues are obtained, it is possible to calculate mass and stiffness stochastic elementary matrices as presented below:

$$\mathbf{M^{(e)}}(\theta) = \mathbf{M^{(e)}} + \sum_{r=1}^{n} \bar{\mathbf{M}}_{\mathbf{r}}^{\mathbf{(e)}}\xi_r(\theta) \tag{3}$$

$$\mathbf{K^{(e)}}(\theta) = \mathbf{K^{(e)}} + \sum_{r=1}^{n} \bar{\mathbf{K}}_{\mathbf{r}}^{\mathbf{(e)}}\xi_r(\theta) \tag{4}$$

Following finite elements theory, one can formulate:

$$\bar{\mathbf{M}}_{\mathbf{r}}^{\mathbf{(e)}} = \xi_r(\theta) \iint_{L_x L_y} \sqrt{\lambda_r} f_r(x,y)\mathbf{N}(x,y)^T \rho \mathbf{N}(x,y)dydx \tag{5}$$

$$\bar{\mathbf{K}}_{\mathbf{r}}^{\mathbf{(e)}} = \xi_r(\theta) \iint_{L_x L_y} \sqrt{\lambda_r} f_r(x,y)\mathbf{B}(x,y)^T \mathbf{HB}(x,y)dydx \tag{6}$$

where L_x and L_y are the correlation lengths, $\mathbf{N}(x,y)$ is the shape functions vector used in FEM formulation, $\mathbf{B}(x,y)$ is a differential operators matrix applied to shape functions (following strength of materials theories), ρ is the mass density and $\mathbf{H}$ is the mechanical properties matrix.

Finally, mass and stiffness global stochastic matrices are determined by concatenation of elementary matrices following nodes and degrees of freedom connectivity:

$$\mathbf{M}(\theta) = \bigcup_{i=1}^{N} \mathbf{M}_{\mathbf{i}}^{(\mathbf{e})}(\theta) \tag{7}$$

$$\mathbf{K}(\theta) = \bigcup_{i=1}^{N} \mathbf{K}_{\mathbf{i}}^{(\mathbf{e})}(\theta) \tag{8}$$

where N is the number of elements.

It is possible to formulate the movement equation for a system including proportional structural damping as follows:

$$\mathbf{M}(\theta)\ddot{\mathbf{U}} + \mathbf{C}(\theta)\dot{\mathbf{U}} + \mathbf{K}(\theta)\mathbf{U} = \mathbf{F} \tag{9}$$

where $\mathbf{C}(\theta) = \beta\mathbf{K}(\theta)$ is the damping matrix, β is the damping coefficient, $\mathbf{U}$ is the displacement vector and $\dot{\mathbf{U}}, \ddot{\mathbf{U}}$ are its first and second time derivatives, respectively.

Applying Fourier's transformation to Eq. (9):

$$\mathbf{U}(\omega,\theta) = \left[-\omega^2\mathbf{M}(\theta) + j\omega\mathbf{C}(\theta) + \mathbf{K}(\theta) \right]^{-1} \mathbf{F}(\omega) \tag{10}$$

A frequency response function (FRF) is defined as the relation between an output and an input in Fourier's domain. For this case, the relation between displacement $\mathbf{U}(\omega,\theta)$ and applied loading $\mathbf{F}(\omega)$ is obtained. The stochastic dynamic system FRF is calculated by:

$$\mathbf{G}(\omega,\theta) = \left[-\omega^2\mathbf{M}(\theta) + j\omega\mathbf{C}(\theta) + \mathbf{K}(\theta) \right]^{-1} \tag{11}$$

As seen in Eq. (11), matrices inversions for each frequency step ω are necessary to evaluate the FRF. However, computational time expended depends on the size of the model, which is proportional to the number of degrees of freedom.

3   ROBUST CONDENSATION PROCEDURE

Real-world engineering models are composed by structures usually containing many thousands or even millions of degrees of freedom. A major difficulty in these cases is the high computational cost involved in matrices operations as the one shown in Eq. (11). However it is possible to obtain responses in a faster way by applying model condensation procedures. The aim of these methods is to apply component-mode synthesis (Craig, 2006) in order to set a basis that represents the dynamic behavior of the nominal model. When applied to mass and stiffness matrices, this basis reduce the effective number of degrees of freedom, decreasing the computational burden and storage memory required.

The first assumption made is that the exact responses provided by the full model can be approached by projections on a reduced vector basis in an alternative subspace as shown in Eq. (12).

$$\mathbf{U}_r = \mathbf{TU} \tag{12}$$

where the matrix $\mathbf{T} \in C^{NxNR}$ is the reduction basis, and $\mathbf{U}_r \in C^{NR}$ with $NR << N$, being NR the number of component-modes retained in the basis.

Mass and stiffness matrices are then pre and post multiplied by this basis, generating a complete new system with similar dynamic behavior:

$$\mathbf{M}_R = \mathbf{T}^T \mathbf{MT} \tag{13}$$

$$\mathbf{K}_R = \mathbf{T}^T \mathbf{KT} \tag{14}$$

where $\mathbf{M}_R$ and $\mathbf{K}_R$ are the reduced mass matrix and reduced stiffness matrix, respectively. It may be also convenient to define a reduced structural damping matrix as described before $\left(\mathbf{C}_R = \beta \mathbf{K}_R \right)$.

Gèrges (2013) studied some techniques to obtain the reduction basis $\mathbf{T}$, concluding that, for linear structures with well-known physical characteristics, it is appropriate to use Ritz modal basis (noted here as ϕ_0) enriched with static residue of a unitary load ($F = 1$) as follows:

$$\bar{\mathbf{U}} = \mathbf{K}^{-1}\mathbf{F} \tag{15}$$

Ritz basis is composed by the nominal system's first vibration mode shapes, resulting in a vector set capable of representing the structural behavior with precision. According to Rosa and de Lima (2016), it is usual to perform analysis in a frequency range that is 1.5 times the maximum frequency desired and to compose Ritz basis with all vibration mode shapes inside the expanded range. Classically, ϕ_0 is obtained through the solution of the following eigenvalue and eigenvector problem of the nominal system:

$$\left[\mathbf{K} - \lambda_i\mathbf{M}\right]\phi_i = \mathbf{0} \tag{16}$$

It is now possible to define the standard Ritz basis $\mathbf{T}_0$:

$$\mathbf{T}_0 = \left[\phi_0 \ \ \bar{\mathbf{U}}\right] \tag{17}$$

The whole process is summarized in Fig. 1.

With this approach, it is expected computational gain when performing matrix inversions such as shown before in Eq. (11). On the other hand, solving stochastic problems requires several evaluations of the entire process. Avoiding the calculation of eigenvalue and eigenvector problems for each one of the samples is a reasonable way to decrease computational efforts even more.

De Lima et al. (2010b) proposed the formulation of a robust condensation basis for viscoelastically damped structures that is updated every time the system undergoes changes. The main contribution of this work is to extend this method to stochastic systems, considering parameter fluctuations as structural modifications. It is possible to express system global matrices as shown in Eqs. (18) and (19).

$$\mathbf{M}(\theta) = \bar{\mathbf{M}} + \Delta\mathbf{M} \tag{18}$$

$$\mathbf{K}(\theta) = \bar{\mathbf{K}} + \Delta\mathbf{K} \tag{19}$$

where $\bar{\mathbf{M}}$, $\bar{\mathbf{K}}$ refer to the system nominal (mean) properties and $\Delta\mathbf{M}$, $\Delta\mathbf{K}$ denote variation on these values.
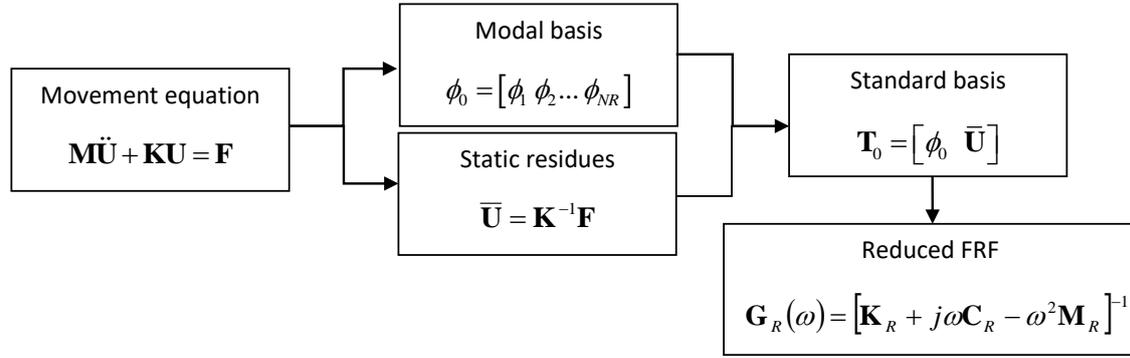
Figure 1. Flowchart of Ritz basis composition process

Equation (20) represents the movement equation of an undamped stochastic system when introducing both Eqs. (18) and (19).

$$\left(\bar{\mathbf{M}} + \Delta\mathbf{M}\right)\ddot{\mathbf{U}} + \left(\bar{\mathbf{K}} + \Delta\mathbf{K}\right)\mathbf{U} = \mathbf{F} \tag{20}$$

In order to keep the left side of the equation only referring to the nominal system, the following manipulation can be done:

$$\bar{\mathbf{M}}\ddot{\mathbf{U}} + \bar{\mathbf{K}}\mathbf{U} = \mathbf{F} - \mathbf{F}_{\Delta\mathbf{M}} - \mathbf{F}_{\Delta\mathbf{K}} \tag{21}$$

where $\mathbf{F}_{\Delta\mathbf{M}} = \Delta\mathbf{M}\ddot{\mathbf{U}}, \mathbf{F}_{\Delta\mathbf{K}} = \Delta\mathbf{K}\mathbf{U}$ are vectors of no-exact forces associated to the modifications.

Since this dynamic behavior and system responses are not known, these vectors cannot be computed exactly. According to Masson et al. (2006) the robust condensation processes consists first in generating a pre-evaluation of no-exact forces to represent the subspace containing them. These vectors are then introduced in standard basis to represent the dynamic behavior of the modified model. Finally, these two steps are repeated for each sample subjected to parameter fluctuations.

Being a system subjected to structural modifications on a parameter p_i, it is possible to formulate a vector of pre-evaluation forces as:

$$\mathbf{F}_{\Delta p_i} = \begin{bmatrix} \mathbf{F}_{\Delta p_i}^{\mathbf{M}} & \mathbf{F}_{\Delta p_i}^{\mathbf{K}} \end{bmatrix} \tag{22}$$

where $\mathbf{F}_{\Delta p}^{\mathbf{M}} = \Delta\mathbf{M}\phi_0\Lambda_0$, $\mathbf{F}_{\Delta p}^{\mathbf{K}} = \Delta\mathbf{K}\phi_0$ are associated to modification in mass and stiffness matrices and $\Lambda_0 = diag\left(\lambda_1,...,\lambda_{NR}\right)$ is a eigenvalue vector.

Now one can calculate the static residues associated to force modifications caused by each parameter p_i in a similar way to that shown before in Eq. (15):

$$\mathbf{R}_{\Delta p_i} = \bar{\mathbf{K}}^{-1} \mathbf{F}_{\Delta p_i} \tag{23}$$

The final step of the robust condensation procedure is to introduce these residues into the standard Ritz basis, generating what will be called *stochastic reduction basis* $\mathbf{T}_S$:

$$\mathbf{T}_S = \begin{bmatrix} \mathbf{T}_0 & \mathbf{R}_\Delta \end{bmatrix} \tag{24}$$

where $R_\Delta = \begin{bmatrix} R_{\Delta p_1} R_{\Delta p_2} ... R_{\Delta p_{np}} \end{bmatrix}$, and *np* is the number of parameters subjected to random effects.

Although reduced mass and stiffness matrices in this case presents larger dimensions, drastic reduction of the computational time is expected owing to only one solution of the eigenvalue and eigenvector problem of the nominal system. For each sample, the basis is re-updated based on the iterative process shown in Fig. 2.
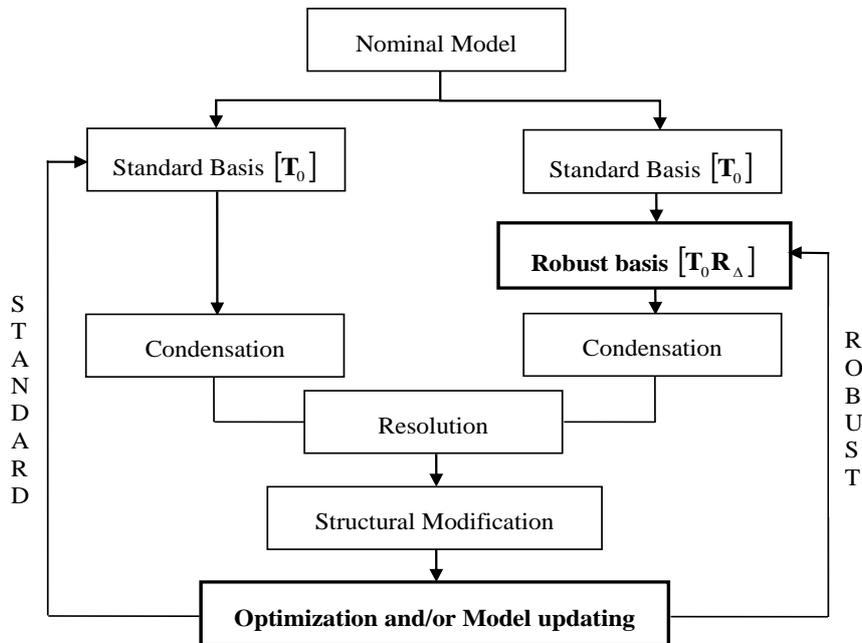
Figure 2. Block-diagram of the condensation robust process (Extract of de Lima et al., 2010)

4   NUMERICAL APPLICATION

A SFEM numerical application was performed and is presented in this section comparing three cases: (a) full model, (b) condensed model through Ritz standard basis $\left(\mathbf{T}_0 \right)$ and (c) condensed model through robust stochastic basis $\left(\mathbf{T}_S \right)$. Monte Carlo Simulation (MCS) was used as stochastic solver (Rubinstein, 1981). All three cases considered thickness variations

and had samples generated by Latin Hypercube (LHC) sampling (Florian, 1992) with 10% uncertainty level over a three-sigma deviation from the mean.

Simulations consist in obtaining FRF envelopes due to a load applied on the middle node of the free boundary edge and displacement was measured in the same point. The desired frequency band is [0-200 Hz]. Following recommendations, this band is expanded to [0-300 Hz] and Ritz standard basis is composed with all vibration mode shapes inside it. For this particular case, it will be composed by 8 eigenvectors plus one static residue.

As no-exact forces vector's dimensions also depends on the number of eigenvectors both for mass and stiffness, robust basis has Ritz nominal basis dimension plus 16 stochastic residues from the modified structure.

In order to estimate an optimal number of samples (n_s) and to validate results, a convergence analysis via root-mean-square deviation (RMSD) of the nominal system's FRFs was performed. Its formulation is presented below:

$$RMSD(n_s) = \sqrt{\frac{1}{n_s} \sum_{i=1}^{n_s} \left| \mathbf{G}(\omega) - \bar{\mathbf{G}}(\omega) \right|^2} \tag{25}$$

After applying Eq. (25), RMSD is normalized by its mean and shown in Fig. 3. At about $n_s = 250$ samples a satisfactory convergence is noted, concluding that this is an optimal number for LHC sampling.
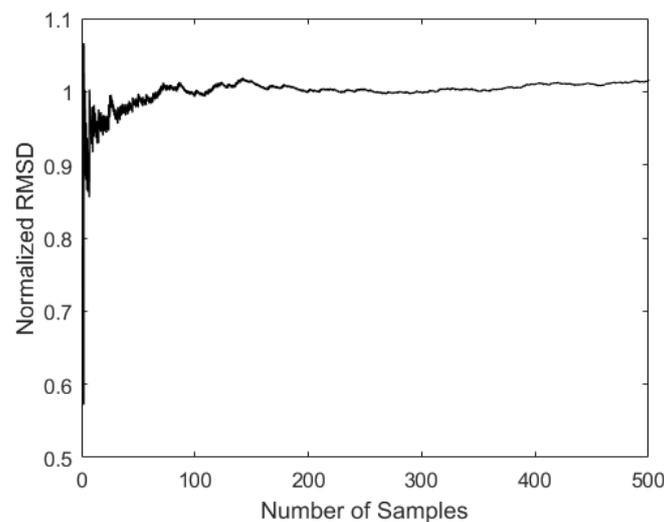
Figure 3. Convergence analysis via RMSD for the nominal system

Figure 4(a) shows the displacement FRF envelopes computed by either the full model and both reduction basis. The maximum difference noticed between the full model and standard Ritz basis results was 0.17% and for robust stochastic basis the maximum difference stood around 0.69%. Despite the first error percentage been lower, both approaches presented excellent results. Seeing these differences is very difficult only analyzing Fig. 4(a). Therefore, a zoom was applied in a small rectangle of envelope's upper boundary and is shown in Fig. 4(b).
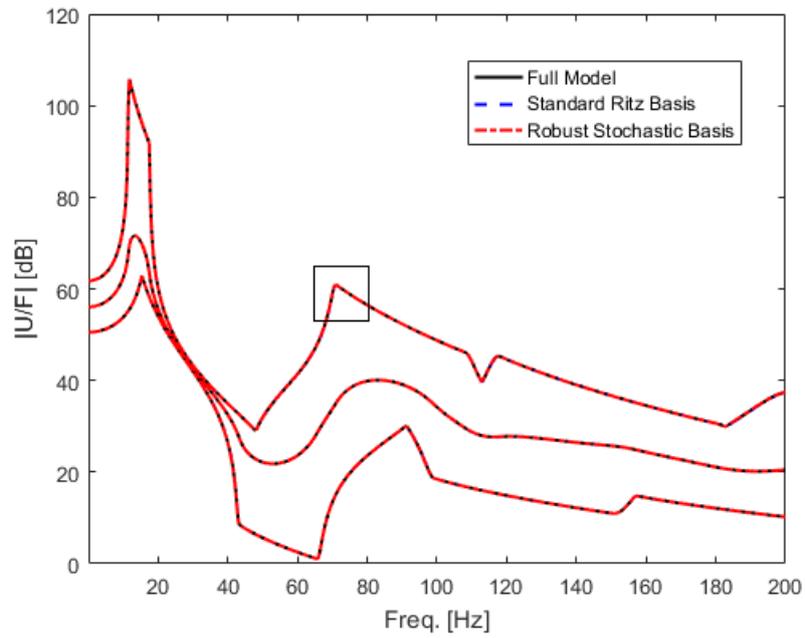
(a)
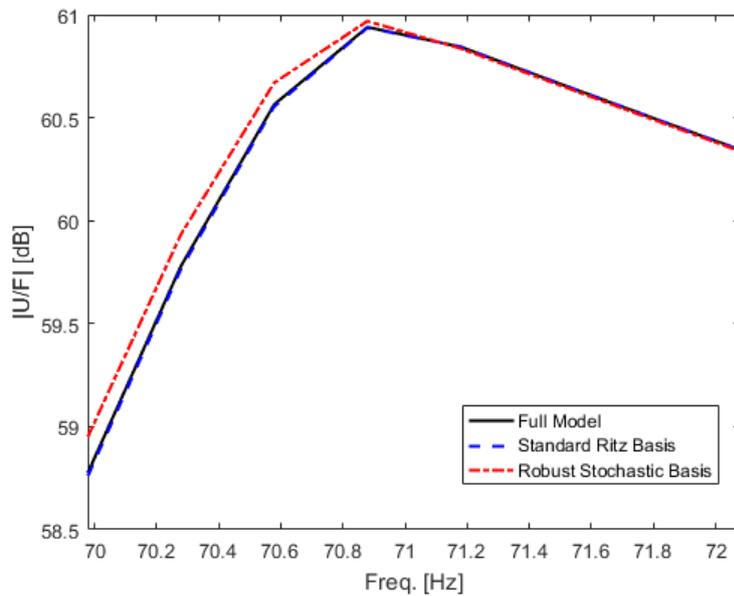
(b)

Figure 4. Comparison of results between the full model, standard Ritz basis and robust stochastic basis (a) Displacement FRF envelopes. (b) Rectangular zoom area of the upper boundary

In the sequence, it was analyzed computational time dispended during FRFs calculation in the complete stochastic process concerning 500 samples. Results are summarized in Table 2. As expected, robust basis results carry less calculation efforts despite having larger

dimensions. As mentioned before, solving the eigenvalue and eigenvector problem causes a great computational burden, and is needed for each sample when applying standard Ritz basis but not for the robust method. When comparing results between both reduction methods, a computational gain of 87.8% is noticed.

Table 2. Model dimensions and computational gain comparison

	Full model	Standard Ritz basis	Robust stochastic basis
Dimensions	605×605	9×9	25×25
Computational gain	-	70.4%	96.4%

5   CONCLUDING REMARKS

A strategy for uncertainty propagation based on stochastic finite element concerning Karhunen-Loève decomposition combined with a robust model reduction technique has been suggested. The main objective is to analyze computational gain of a technique first developed for different approaches (viscoelastically damped structures) and after expanded for the stochastic context described herein. Numerical simulations showed that the robust model condensation strategy is as efficient as standard Ritz basis enriched with static residues to approach full model FRF envelopes. They have provided accurate and effective results with a low computational cost, mainly generated by the drastic reduction of the number of DOFs, which suggested that the robust basis is well adapted to be applied to stochastic models.

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