Computation of Moments in the Anisotropic Plane Elasticity Fast Multipole formulation

Afonso Dias Jr

afonso.56@hotmail.com

Universidade of Brasilia - Departamento do Engenharia Mecanica

Darcy Ribeiro Campus, 70910-900, Brasilia, Brazil,

Eder Lima Albuquerque

eder@unb.br

Universidade de Brasilia - Departamento de Engenharia Mecanica

Darcy Ribeiro Campus, 70910-900, Brasilia, Brazil,

Adriana dos Reis

adriana@fem.unicamp.br

Abstract. *In this work we will present the computation of moments in the anisotropic plane elasticity fast multipole formulation. Fundamental solutions of plane elasticity are represented by complex functions from the classical 2D elasticity theory. The Multipole Expansion for kernels U (displacement field) and T (traction field) will be computed using Taylor series expansion. The convergence of the series expansion to the fundamental solutions is analyzed considering different numbers of series terms and different distance from the source point to the field point. Moments will be used to evaluate integrals of influence matrices when elements are far away from the source point, whereas the conventional approach will be applied to evaluate the integrals in order to compare results obtained by the multipole expansion.*

Keywords: *Fast Multipole Method, Boundary Element Method, Anisotropic plane elasticity*

1 Introduction

Currently, the use of composite materials have long been used by different fields of science. The reasons for the large utilization are their mechanical properties. As an example, they present good performance in extreme temperatures (cold or hot) and the biggest specific stiffness and specific strength if compared to other materials. Futher more, there are many projects that require specific mechanics features, as high stiffness, or mechanical strength , low density that can be obtained only if composite materials are used.

Over the years, with the developing of composite materials, many projects have considered their use. Then, it was necessary to investigate the mechanical behaviour of these materials, studing their stress and the displacement. As composite materials are anisotropic materials, the analitical solutions are very hard to be obtained.

A numerical method used to study the mechanical behaviour of composite materials is the Boundary Element Method (BEM). [Katsikadelis(2002)], [Beer(2008)], [Aliabadi(2002)], [Gaul(2002)], [Banerjee(1994)] are some books where it is possible to find more information about it.

One of the disadvantages of the BEM is its inefficiency to solve large-scale problems. The matrices produced by this method are dense and non-symmetric. With this method it is necessary O(N^2) operations to compute the coefficients of the matrices and O(N^3) operations to solve the system of equations by direct solvers, being N the number of equations in the linear system of DOFs. To solve this drawback, we can use Fast Multipole Method (FMM) to accelerate the solutions of BEM. As consequence, we will find a reduction in the CPU time and a reduction in the memory used to produce the matrices and solve the system of equations. The union between BEM and FMM is known as fast multipole BEM or simply fast BEM. In recent years the use of fast multipole BEM was investigated by many authors, including [Peirce(1995)] for 2D elastostatics, [Popov(2001)] for 3D elastostatics, [Yoshida(2001a)] and [Yoshida(2001b)] for 3D elastostatic crack problems, [Liu(2005)] for the modelling of carbon-nanotube composites, [Wang(2005), Wang(2004)] for the simulation of composite materials and [Wang(2006)] for the analysis of fatigue crack growth. A recent review on the fast multipole BEM can be found in [Nishimura(2002)]. The main idea of the FMM is to change the node-to-node (or element-to-element) interactions to cell-to-cell interactions. With the expansion of fundamental solutions in series and the grouping of elements into cells, it will reduce the computational cost of the BEM.

In this paper, the operations of FMMBEM for 2D anisotropic elasticity problems is presented. The fundamental solution for anisotropic problems will be expanded. After, it will be introduced the operations *Moment*, *Moment to Moment*, *Local* and *Local to Local* to evaluate the integrals founded in the BEM. A hierarchical tree structure will be used in this step. The tree is used to group the elements in cells. Finally, we will compare the results of integrals produced by the BEM with results of integrals produced by fast the BEM to analyse the influence of the number of terms in the Taylor series on the accuracy of the integration. The anisotropic formulation used in this simulation will be modeled by Lekhnitskii formalism.

2 Expansion of Fundamental Solution

The fundamental solutions for anisotropic problems is given by:

$$U_{ji}(\mathbf{z},\mathbf{z_o}) = 2Re[q_{i1}\Phi_1 + q_{i2}\Phi_2] = 2Re[q_{i1}A_{j1}\ln(z_{o1}-z_1) + q_{i2}A_{j2}\ln(z_{o2}-z_2)] \quad (1)$$

and

$$T_{ij}(\mathbf{z},\mathbf{z_o}) = 2Re\left[\frac{g_{j1}(\mu_1 n_1 - n_2)A_{i1}}{(z_{o1}-z_1)} + \frac{g_{j2}(\mu_2 n_1 - n_1)A_{i2}}{(z_{o2}-z_2)}\right] \quad (2)$$

where the terms μ_k, q_{ik}, g_{jk} and A_{ik} are given:

$$a_{11}\mu^4 - 2a_{16}\mu^3 + (2a_{12}+a_{66})\mu^2 - 2a_{26}\mu + a_{22} = 0, \quad (3)$$

$$q_{ik} = \begin{bmatrix} a_{11}\mu_k^2 + a_{12} - a_{16}\mu_k \\ a_{12}\mu_k + a_{22}/\mu_k - a_{26} \end{bmatrix} \quad (4)$$

$$[g_{jk}] = \begin{bmatrix} \mu_1 & \mu_2 \\ -1 & -1 \end{bmatrix} \quad (5)$$

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ \mu_1 & -\overline{\mu}_1 & \mu_2 & -\overline{\mu}_2 \\ q_{11} & -\overline{q}_{11} & q_{12} & -\overline{q}_{12} \\ q_{21} & -\overline{q}_{21} & q_{22} & -\overline{q}_{22} \end{bmatrix} \begin{Bmatrix} A_{j1} \\ \overline{A}_{j1} \\ A_{j2} \\ \overline{A}_{j2} \end{Bmatrix} = \begin{Bmatrix} \delta_{j2}/(2\pi i) \\ -\delta_{j1}/(2\pi i) \\ 0 \\ 0 \end{Bmatrix} \quad (6)$$

All terms above (μ_k, q_{ik}, g_{jk} and A_{ik}) are material complex constants. Their values are presented in the Table 1. In the equation (3), the terms a_{11}, a_{12}, a_{16}, a_{22}, a_{26} and a_{66} are the coefficients of elastic compliance.

The derivation of equation (3) and the equation (6) can be found in [Sollero(1994)] and [Albuquerque(2001)]:

Table 1: The values of material complex constants.

i and j	μ	A_{ij}	q_{ij}	g_{ij}
11	-0.1623 + 0.8860i	0.0431 - 02301i	-0.1411 - 0.0276i	-0.1623 + 0.8860i
12	0.1446 + 0.8440i	0.0264 + 0.2301i	-0.1329 + 0.0263i	0.1146 + 0.8440i
21	——	0.1969 - 0.0041i	-0.0015 - 0.1209i	-1.000
22	——	-0.1972 - 0.0755i	0.069 - 01233i	-1.000

The field point (z) and the source point (z_o) are represented by:

$$\mathbf{z} = \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} x_1 + \mu_1 x_2 \\ x_1 + \mu_2 x_2 \end{Bmatrix} \quad (7)$$

$$\mathbf{z} = \left\{ \begin{array}{c} z_{o1} \\ z_{o2} \end{array} \right\} = \left\{ \begin{array}{c} x_{o1} + \mu_1 x_{o2} \\ x_{o1} + \mu_2 x_{o2} \end{array} \right\} \tag{8}$$

To exapand the fundamental solution, we will introduced the function:

$$G(z_o, z) = \log(z_o - z) \tag{9}$$

and the derivative of the equation (9):

$$G^{'}(z_o, z) = \frac{\partial G(z_o, z)}{\partial z} = \frac{1}{(z - z_o)} \tag{10}$$

We will rewrite the equations (1) and (2) as:

$$U_{ij}(z_o, z) = 2Re\left[q_{i1} A_{j1} G(z_{1o}, z_1) + q_{i2} A_{j2} G(z_{2o}, z_2)\right] \tag{11}$$

$$T_{ij}(z_o, z) = 2Re\left[G^{'}(z_{1o}, z_1) g_{i1}(\mu_1 n_1 - n_2) A_{j1} + G^{'}(z_{o2}, z_2) g_{i2}(\mu_2 n_1 - n_2) A_{j2}\right] \tag{12}$$

Now, a intermediate point (z_{c_i}) will be introduce in the equation (9):

$$z_{c_i} = \left[\begin{array}{c} z_{c_1} \\ z_{c_2} \end{array} \right] = \left[\begin{array}{c} x_c + \mu_1 y_c \\ x_c + \mu_2 y_c \end{array} \right] \tag{13}$$

$$G(z_o, z) = \log(z - z_{c_i} - z_o + z_{c_i}) \tag{14}$$

Using the results from [Reis(2013)], we can find the final expression for the equation (14):

$$G(z_o, z) = \sum_{k=0}^{\infty} O_k(z_o - z_{k_c}) I_k(z - z_{k_c}) \tag{15}$$

where:

$$O_k(z) = \frac{(k-1)!}{z^k}, \;\; for \; k > 1, \tag{16}$$

$$O_o(z) = -\log(z) \tag{17}$$

$$I_k(z) = \frac{z^k}{k!}, \;\; for \; k > 0. \tag{18}$$

For the equation (10), we can find the following expression:

$$G^{'}(z_o, z) = \frac{\partial G(z_o, z)}{\partial z} = \frac{1}{(z - z_o)} = \sum_{k=1}^{\infty} O_k(z_o - z_{k_c}) I_{k-1}(z - z_{k_c}) \tag{19}$$

The complete derivation of equations (15) and (19) can be found in [Reis(2013)]. In this paper, our focus will be in the operations of FMM.

3 Multipole Expansion the U kernel Integral

To do the multipole expansion, consider two points, (z_{c_1},z_{c_2}), near the field points (z_1,z_2). So, we will consider that the distance between $\mid z_1 - z_{c_1} \mid$ and $\mid z_2 - z_{c_2} \mid$ are less than $\mid z_{1o} - z_{c_1} \mid$ $\mid z_{2o} - z_{c_2} \mid$. So, $\mid z_1 - z_{c_1} \mid \ll \mid z_{1o} - z_{c_1} \mid$ and $\mid z_2 - z_{c_2} \mid \ll \mid z_{2o} - z_{c_2} \mid$. Using the new auxiliary function $I_k(z)$ e $O_k(z)$ we have:

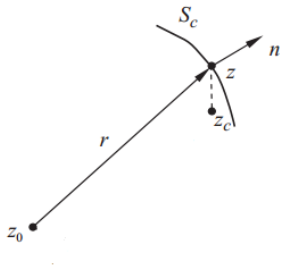

Figure 1: Mutilpole expansion around z_c

$$\int_\Gamma t_j U_{ij}(\mathbf{z_o}, \mathbf{z}) d\Gamma = \int_\Gamma t_j (2Re\left[q_{i1} A_{j1} G(z_{1o}, z_1) + q_{i2} A_{j2} G(z_{2o}, z_2)\right]) d\Gamma \tag{20}$$

$$\int_\Gamma t_j U_{ij}(\mathbf{z_o}, \mathbf{z}) d\Gamma = 2Re \int_\Gamma t_j q_{i1} A_{j1} G(z_{1o}, z_1) d\Gamma + 2Re \int_\Gamma t_j q_{i2} A_{j2} G(z_{2o}, z_2) d\Gamma \tag{21}$$

To decrease the number of equation, we will work with the first integral in the right side of equation (21) and we will just expand the result for the second integral in the left side of equation (21). So, using the equation (15), we will write the following equation:

$$2Re \int_\Gamma t_j [q_{i1} A_{j1} G(z_{1o}, z_1)] d\Gamma = 2Re \int_\Gamma t_j q_{i1} A_{j1} \left(\sum_{k=0}^{\infty} O_{k_1}(z_{1o} - z_{c1}) I_{k_1}(z_1 - z_{c_1}) \right) d\Gamma \tag{22}$$

$$2Re \int_\Gamma t_j [q_{i1} A_{j1} G(z_{1o}, z_1)] d\Gamma = 2Re \sum_{k=0}^{\infty} O_{k_1}(z_{1o} - z_{c_1}) M_{k_1}(z_{c1}) \tag{23}$$

where:

$$M_{k_1}(z_{c1}) = \int_{S_c} t_j q_{i1} A_{j1} I_{k_1}(z_1 - z_{c_1}) dS \;\; k = 0, 1, 2, ... \tag{24}$$

Equation (24) is called *multipole expansion*. This equation can be readly evaluated , because there is no dependence on the source point (z_o). So, for any position of z_o away from S_c, this term no longer needs to be computed. As consequence, the number of operations decrease. According to [Braga(2012)], this process is the key of the FMM. The same equation can be obtained for the other integral:

$$2Re \int_\Gamma t_j [q_{i2} A_{j1} G(z_{2o}, z_2)] d\Gamma = 2Re \sum_{k=0}^{\infty} O_{k_2}(z_{2o} - z_{c_2}) M_{k_2}(z_c) \tag{25}$$

where:

$$M_{k_2}(z_{c2}) = \int_{S_c} t_j q_{i2} A_{j2} I_{k_2}(z_2 - z_{c2}) dS \;\; k = 0, 1, 2, ... \tag{26}$$

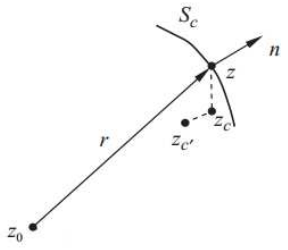

Figure 2: Translation z_c for $z_c^{'}$

3.1 Moment to Moment Translation (M2M)

To present the operation *Moment to moment* opearation, consider that the points (z_{c1}) and (z_{c2}) were moved to new positions $(z_{c'1})$ and $(z_{c'2})$. It is necessary to obtain a new moments without recomputing the equations (24) and (26). Consider the property below:

$$I_k(z_1 + z_2) = \sum_{l=0}^{k} I_{k-l}(z_1) I_l(z_2) = \sum_{l=0}^{k} I_l(z_1) I_{k-l}(z_2) \tag{27}$$

so:

$$M_{k_1}(z_{c'1}) = \int_{\S_c} t_j q_{i1} A_{j1} I_{k_1}(z_1 - z_{c'1}) dS \tag{28}$$

$$M_{k_1}(z_{c'1}) = \int_{\S_c} t_j q_{i1} A_{j1} I_{k_1}[(z_1 - z_{c1}) + (z_{k_{c1}} - z_{k_{c'1}})] dS \tag{29}$$

After some mathematics manipulations, we will find:

$$M_{k_1}(z_{c'1}) = \sum_{l=0}^{k} I_{k_1-l}(z_{k_{c1}} - z_{k_{c'1}}) M_{l_1}(z_c) \tag{30}$$

$$M_{l_1}(z_c) = \int_{S_c} t_j q_{i1} A_{j1} I_{l_1}(z_1 - z_{c1}) dS \tag{31}$$

the same results can be obtained for equation (26):

$$M_{k_2}(z_{c'2}) = \sum_{l=0}^{k} I_{k_2-l}(z_{k_{c2}} - z_{k_{c'2}}) M_{l_2}(z_c) \tag{32}$$

$$M_{l_2}(z_c) = \int_{\S_c} t_j q_{i2} A_{j2} I_{l_2}(z_2 - z_{c2}) dS \tag{33}$$

3.2 Local expansion and Moment to Local Translation (M2L)

To present the last two operations, consider a point (z_{L1}) and (z_{L2}) next to the source point (z_{o1}) and (z_{o2}). So, considering this aproximation ($\mid z_{o1} - z_{L1} \mid \ll \mid z_1 - z_{L1} \mid$) and ($\mid z_{o2} - z_{L2} \mid \ll \mid z_2 - z_{L2} \mid$),the equation for local expansion is given by:

$$2Re \int_{\Gamma} t_j [q_{i1} A_{j1} G(z_{o1}, z_1)] d\Gamma = 2Re \sum_{k=0}^{\infty} O_{k_1}[(z_{L1} - z_{c1}) + (z_{o1} - z_{k_c})] M_{k_1}(z_{c1}) \tag{34}$$

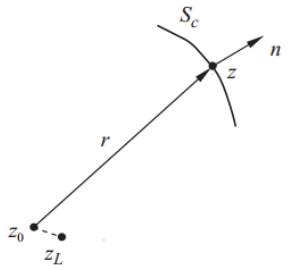

Figure 3: Local expansion around z_L

Using the property below, we will find another equation:

$$O_k(z_1+z_2)=\sum_{l=0}^{\infty}(-1)^l O_{k+l}(z_1)I_l(z_2)\ \ for\ |z_2|<|z_1| \tag{35}$$

Thus:

$$2Re\int_\Gamma t_j[q_{i1}A_{j1}G(z_{o1},z_1)]d\Gamma = 2Re\left[\sum_{k=0}^{\infty}(-1)^l O_{k_1+l_1}(z_{L1}-z_{c1})I_{l_1}(z_{o1}-z_{c1})\right]M_{k_1}(z_{c1}) \tag{36}$$

After some manipulations, the followin equations are obtained:

$$2Re\int_\Gamma t_j[q_{i1}A_{j1}G(z_{o1},z_1)]d\Gamma = 2Re\left[\sum_{l=0}^{\infty}L_{k_1}(z_{L1})I_{l_1}(z_{o1}-z_{c1})\right] \tag{37}$$

where:

$$L_{k_1}(z_{L1})=(-1)^k\left[\sum_{k=0}^{\infty}O_{k_1+l_1}(z_{L1}-z_{c1})I_{l_1}(z_{o1}-z_{c1})\right]M_{k_1}(z_{c1}) \tag{38}$$

Equation (37) is called *local expansion* and the equation (38) is *moment-to-local translation*. Finally, the same results can be obtained for the following equation:

$$2Re\int_\Gamma t_j[q_{i2}A_{j2}G(z_{o1},z_2)]d\Gamma = 2Re\left[\sum_{l=0}^{\infty}L_{k_2}(z_{L2})I_{l_2}(z_{o2}-z_{c2})\right] \tag{39}$$

$$L_{k_2}(z_{L2})=(-1)^k\left[\sum_{k=0}^{\infty}O_{k_2+l_2}(z_{L2}-z_{c2})I_{l_2}(z_{o2}-z_{c2})\right]M_{k_2}(z_{c2}) \tag{40}$$

3.3 Local to local translations

The last translation is the *local to local*. For this situation, consider that the points z_{L1} and z_{L2} were moved to $z^{'}_{L1}$ e $z^{'}_{L2}$. So, the results are:

$$2Re\int_\Gamma t_j[q_{i1}A_{j1}G(z_{o1},z_1)]d\Gamma = 2Re\left[\sum_{l=0}^{\infty}L_{k_1}(z_{L1})I_{l_1}((z_{o1}-z^{'}_{L1})+(z^{'}_{L1}-z_L))\right] \tag{41}$$

$$2Re\int_\Gamma t_j[q_{i1}A_{j1}G(z_{o1},z_1)]d\Gamma = 2Re\sum_{l=0}^{\infty}L_{k_1}(z^{'}_{L1})I_{p_1}(z_{1o}-z^{'}_{L1}) \tag{42}$$

where:

$$L_{k_1}(z^{'}_{L1})=\sum_{k=0}^{p-l}I_{k_1-l_1}(z^{'}_{L1}-z_L)L_{l+k}(z_{l1}) \tag{43}$$

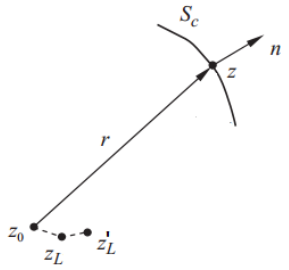

Figure 4: Translation z_L for $z_L^{'}$

Equation (43) is called *local-to-local translation*. Following the same steps, it is possible to obtain:

$$2Re\int_{\Gamma} t_j[q_{i2}A_{j2}G(z_{1o},z_2)]d\Gamma = 2Re\sum_{l=0}^{\infty} L_{k_2}(z_{L2}^{'})I_{p_2}(z_{2o}-z_{L2}^{'}) \tag{44}$$

$$L_{k_2}(z_{L2}^{'}) = \sum_{k=0}^{p-l} I_{k_2-l_2}(z_{L2}^{'}-z_L)L_{l_2+k_2}(z_{L2}) \tag{45}$$

4 Multipole Expansion the T kernel Integral

To avoid the demonstration presented above, it will be showed only the final results for the kernel T. These results are obtained by doing the same steps presented in the section before. So, we have:

$$\int_{\Gamma} u_j T_{ij}(\mathbf{z_o},\mathbf{z})d\Gamma = \\ 2Re\int_{\Gamma} u_j\left[G^{'}(z_{o1},z_1)g_{i1}(\mu_1 n_1 - n_2)A_{j1} + G^{'}(z_{o2},z_2)g_{i2}(\mu_2 n_1 - n_2)A_{j2}\right]d\Gamma \tag{46}$$

$$\int_{\Gamma} u_j T_{ij}(\mathbf{z_o},\mathbf{z})d\Gamma = 2Re\int_{\Gamma} u_j\left[G^{'}(z_{o1},z_1)g_{i1}(\mu_1 n_1 - n_2)A_{j1}\right]d\Gamma + \\ 2Re\int_{\Gamma} u_j\left[G^{'}(z_{o2},z_2)g_{i2}(\mu_2 n_1 - n_2)A_{j2}\right]d\Gamma \tag{47}$$

The *multipole expansion* is:

$$2Re\int_{\Gamma} u_j\left[G^{'}(z_{o1},z_1)g_{i1}(\mu_1 n_1 - n_2)A_{j1}\right]d\Gamma = 2Re\sum_{k=0}^{\infty} O_k(z_{o1}-z_{c_1})\tilde{M}_{k_1}(z_{c1}) \tag{48}$$

where:

$$\tilde{M}_{k_1}(z_{c1}) = \int_{S_c} u_j g_{i1}(\mu_1 n_1 - n_2)A_{j1}I_{k_1-l}(z_1 - z_{c_1})dS \;\; k = 0,1,2,... \tag{49}$$

and:

$$2Re\int_{\Gamma} u_j\left[G^{'}(z_{o2},z_2)g_{i2}(\mu_2 n_1 - n_2)A_{j2}\right]d\Gamma = 2Re\sum_{k=0}^{\infty} O_k(z_{o2}-z_{c_2})\tilde{M}_{k_2}(z_{c2}) \tag{50}$$

$$\tilde{M}_{k_2}(z_{c2}) = \int_{S_c} u_j g_{i2}(\mu_1 n_1 - n_2) A_{j2} I_{k_2-l}(z_2 - z_{c_2}) dS \;\; k = 0, 1, 2, ... \tag{51}$$

4.1 Moment to Moment Translation (M2M)

$$\tilde{M}_{k_1}(z_{c'1}) = \sum_{l=0}^{k} I_{k_1-l}(z_{k_{c1}} - z_{k_{c'1}}) \tilde{M}_{l_1}(z_c) \tag{52}$$

$$\tilde{M}_{l_1}(z_c) = \int_{S_c} t_j g_{i1}(\mu_1 n_1 - n_2) A_{j1} I_{l_1-l}(z_1 - z_{c1}) dS \tag{53}$$

$$\tilde{M}_{k_2}(z_{c'2}) = \sum_{l=0}^{k} I_{k_2-l}(z_{k_{c2}} - z_{k_{c'2}}) \tilde{M}_{l_2}(z_c) \tag{54}$$

$$\tilde{M}_{l_2}(z_c) = \int_{S_c} t_j g_{i2}(\mu_2 n_1 - n_2) A_{j2} I_{l_2-l}(z_2 - z_{c2}) dS \tag{55}$$

4.2 Local expansion and Moment to Local Translation (M2L)

$$2Re \int_{\Gamma} t_j [g_{i1}(\mu_1 n_1 - n_2) A_{j1} G(z_{o1}, z_1)] d\Gamma = 2Re \left[\sum_{l=0}^{\infty} \tilde{L}_{k_1}(z_{L1}) I_{l_1}(z_{o1} - z_{c1}) \right] \tag{56}$$

where:

$$\tilde{L}_{k_1}(z_{1L}) = (-1)^k \left[\sum_{k=0}^{\infty} O_{k_1+l_1}(z_{L1} - z_{c1}) I_{l_1-1}(z_{o1} - z_{c1}) \right] \tilde{M}_{k_1}(z_{c1}) \tag{57}$$

For the other equation, we have:

$$2Re \int_{\Gamma} t_j [g_{i1}(\mu_2 n_1 - n_2) A_{j2} G(z_{o2}, z_2)] d\Gamma = 2Re \left[\sum_{l=0}^{\infty} \tilde{L}_{k_2}(z_{L2}) I_{l_2}(z_{o2} - z_{c2}) \right] \tag{58}$$

$$\tilde{L}_{k_2}(z_{2L}) = (-1)^k \left[\sum_{k=0}^{\infty} O_{k_2+l_2}(z_{L2} - z_{c1}) I_{l_2-1}(z_{o2} - z_{c2}) \right] \tilde{M}_{k_1}(z_{2c}) \tag{59}$$

4.3 Local to local translations

$$2Re \int_{\Gamma} t_j [g_{i1}(\mu_1 n_1 - n_2) A_{j1} G(z_{o1}, z_1)] d\Gamma = 2Re \sum_{l=0}^{\infty} \tilde{L}_{k_1}(z'_{L1}) I_{p_1}(z_{o1} - z'_{L1}) \tag{60}$$

where:

$$\tilde{L}_{1k}(z'_{L1}) = \sum_{k=0}^{p-l} I_{k_1-l_1}(z'_{L1} - z_L) \tilde{L}_{l+k}(z_{l1}) \tag{61}$$

Equation (61) is called *local-to-local translation*. For the other integral, we have:

$$2Re \int_{\Gamma} t_j [g_{i2}(\mu_2 n_1 - n_2) A_{j2} G(z_{o1}, z_2)] d\Gamma = 2Re \sum_{l=0}^{\infty} \tilde{L}_{k_2}(z'_{L2}) I_{p_2}(z_{2o} - z'_{L2}) \tag{62}$$

$$L_{k_2}(z'_{L2}) = \sum_{k=0}^{p-l} I_{k_2-l_2}(z'_{L2} - z_L) L_{l+k}(z_{l_2}) \tag{63}$$

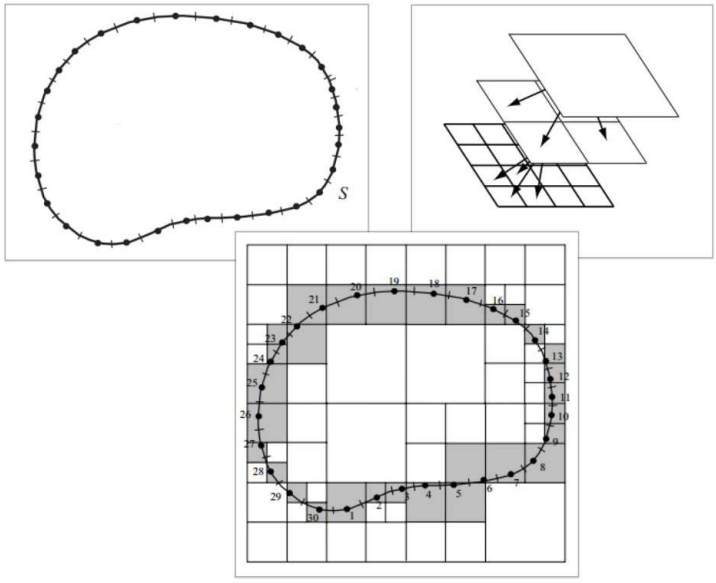

Figure 5: Boundary subdivided by square - fonte: [Liu(2009)]

5 Tree hierarchy strucure

To explain how the operations of the FMM are used, we will introduce the tree hierarchy strucure. First of all, we will circumscribe the boundary by a square. This step is known as level zero. Then, divide this square into four equal pieces. We will create four new squares from the square which circumscribes our boundary. Each square created will be known as cell. This step is known as level 1. Therefore, when we create these two levels, an inter-dependency between the zero level and level 1 is built. The square of the level zero is considered the father of the four square cells of the level 1. We will continue the subdivision of each squares in four new squares, until reach a certain level. The above process will be interrupted when the number of nodes in each cell does not exceed a predetermined number. Finally, we will call leaf cell the cells that are no longer divided.

Now, we will show the operations presented in the previous sections. For the squares known as leaf cells, we will create a center. We will use the *multipole expansion* to transfer the position of the node to the centroid of the leaf cell. Then, we will transfer the position of the centroid in the leaf cell to the centroid of the father cell. After, this centroid will be transfered to his father cell centroid. For this step, we will use the translation $M2M$. We will continue this process until we reach the level 2, carrying the position of these centroides with operation $M2M$. The described process is known as *upward*. The Figure 6 represents the *upward* step.

After the level 2 is reached, we will use the translations $M2L$, *L2L* and the *local expansion*. This step is known as *downward*. First of all, before use the tranlastions $M2L$, *L2L* and the *local expansion*, we must use the subdivison created by tree hierarchy to qualify elements that are near, well separated and distant far from the leaf cell that contains the source point. According to [Liu(2009)], *near cells* are those that have, at least, one vertex in common with the leaf cell. For a cell to be considered *well separated*, we should look at the relationship between the father cell of the leaf cell and the father cell of the other cells. If the father cells are adjacent cells, the cell will be considered *well separated* from the leaf cell. Finally, a cell is considered *far from of the cell* that has the leaf cell if they do not have adjacent fathers cells. Figure 7

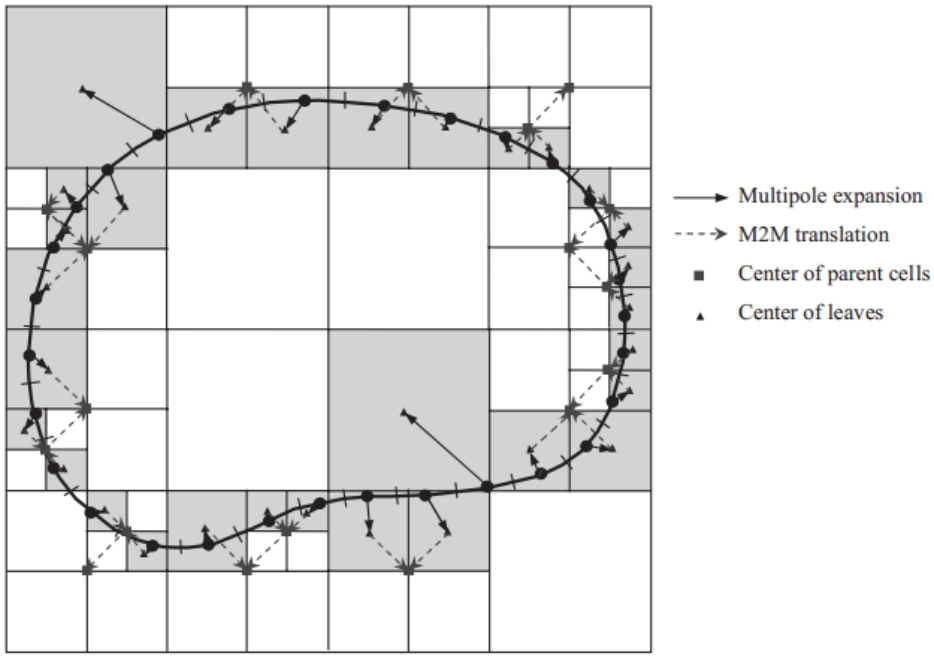

Figure 6: Upward process

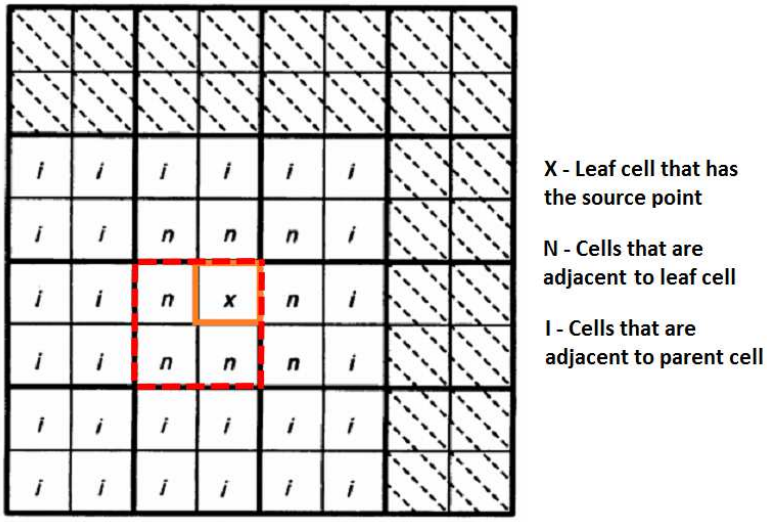

Figure 7: Relation between the cells

illustrates what was described above.

So, after this classification, we will start the *downward* process. The beginning of it is the level 2. We will use the $M2L$ translation on all cells to change the centroid position of *well separated* and cells that are *far from the cell*. The $L2L$ will be used to transfer the position of the centroid of the parent cell to the centroid of the leaf cell. Finally, we will use the *local expansion* to transfer the position of the centroid of cell to the point source.

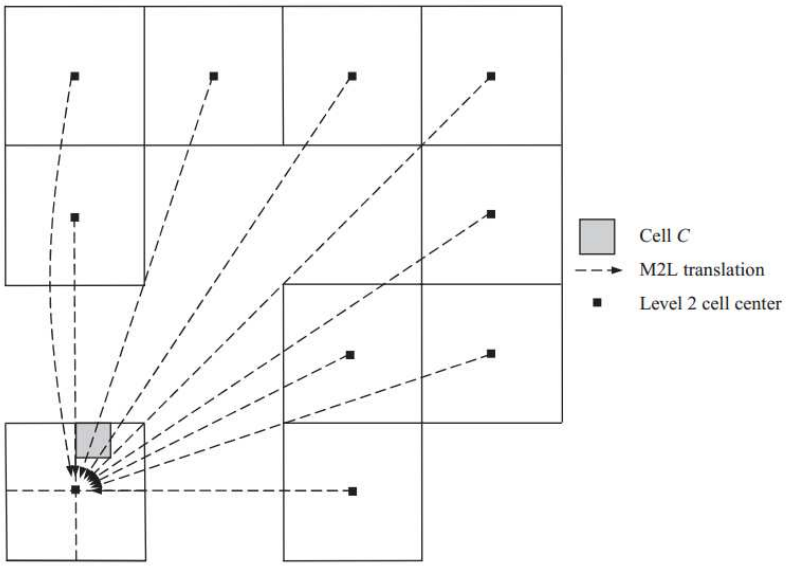

Figure 8: Downward process with M2L translation

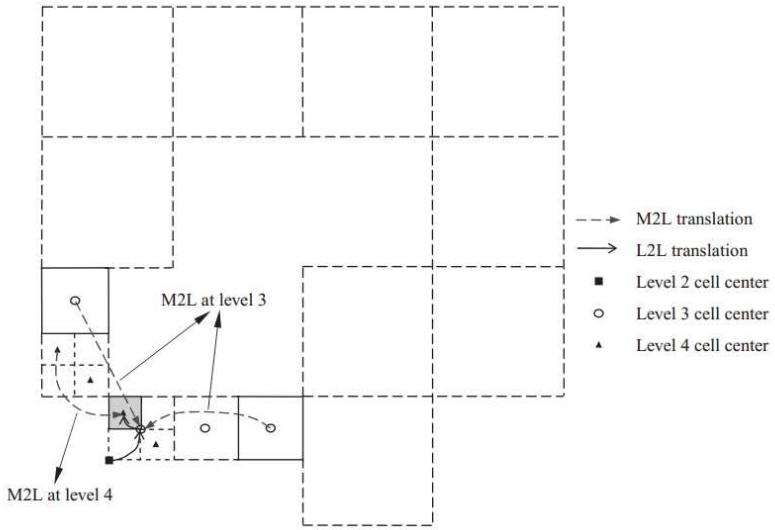

Figure 9: Downward process with M2L translation

6 Numerical Results

This section aims to demonstrate operations of the FMMBEM and also the efficiency of the number of terms in the Taylor series expansion for matrixes $[H]$ and $[G]$. We will do this using elements 15 and 16 presented in the Figure 10. Sources points are nodes 2 and 3. Figure 11 shows the numbering of cells. This problem was assumed the maximum number of elements per cell equal to 1. Figure 12 shows the numbering of the nodes along with the cells. First of all, matrixes $[H]$ and $[G]$ are calculated by using the FMMBEM in the following sequence: calculate the *moments* of the elements 15 and 16 in relation to the center of the cells 54 and 55, respectively (see Fig 11). Then, using *M2M* operation, we will make the translation to the center of cell 37, which is the parent of cells 54 and 55. Then, sum up the two moments,

since both are calculated from the same point (center of cell 37). Then, using *M2L*, compute the local expansion L from the center point of the cell 37 to the center of the cell 20, which is the parent cell of cells 38 and 39. Cells 38 and 39, in turn, contain the sourcers points, which are nodes 2 and 3 (see Figure 12). The translation of expansion of the cell location 20 to the centers of the cells 38 and 39 is next. Thereafter, using *local translation*, calculate integrals of matrixes $[G]$ and $[H]$. Finally, to compare the results, we compute matrices $[G]$ and $[H]$ using the standard boundary element method. Table 2 shows a comparison of the FMMBEM with different number of terms in the series expansion with standard BEM.

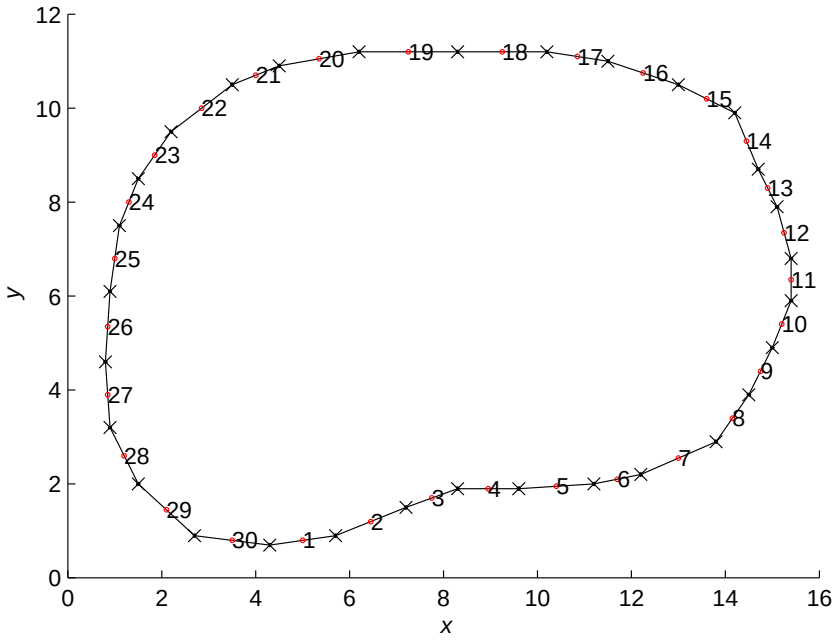

Figure 10: Element mesh with constant number of nodes.

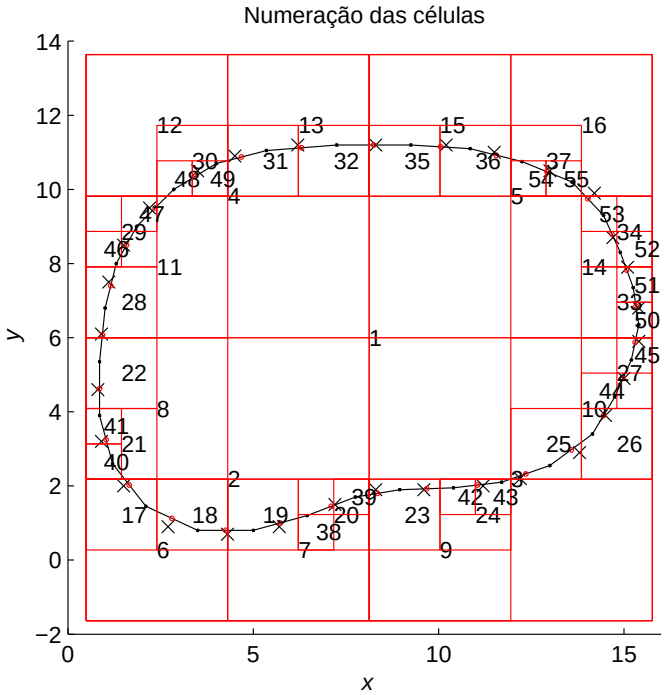

Figure 11: Numbering of cells.

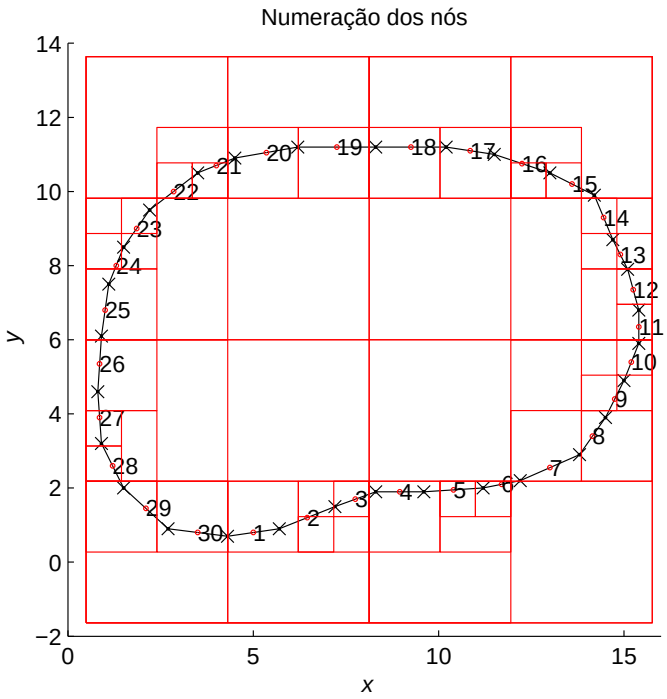

Figure 12: Cells and number of nodes.

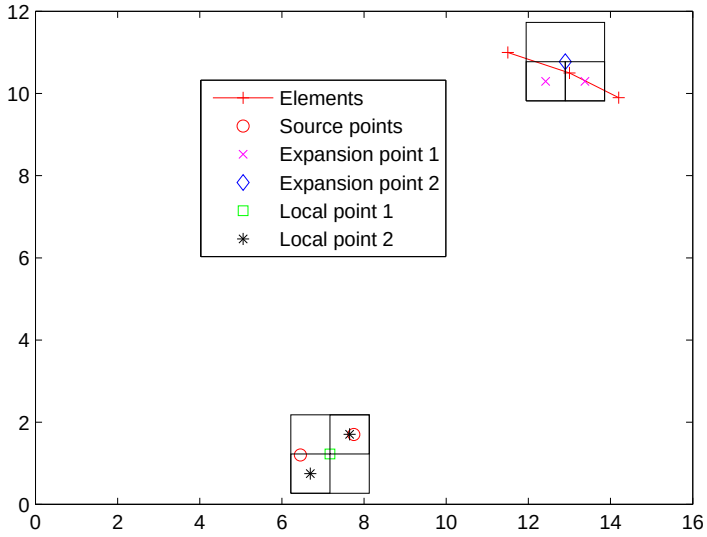

Figure 13

Table 2: Comparison between FMM and BEM

Number of terms	$[G]$ node 2	$[H]$ node 2	$[G]$ node 3	$[H]$ node 3
1	-0.38176	0.029265	-0.37402	0.03004
2	-0.38036	0.033828	-0.37136	0.035655
4	-0.3804	0.034124	-0.37128	0.036273
6	-0.3804	0.034112	-0.37128	0.036256
8	-0.3804	0.034112	-0.37128	0.036256
BEM	$-0.3804i$	0.034112	-0.37128	0.036256

As it can be seen, the agreement between FMMBEM and BEM is very good even with few terms in the series expansion. The agreement improve with the increase of terms on the Taylor series. Furthermore, for the same accuracy, matrix H demands a higher number of terms in the Taylor series that matrix G. This can be explained by the order of singularity of matrix H that is higher than the order of matrix G.

6.1 Conclusion

This paper showed the influence of number of terms of the Taylor series in the operations of the FMMBEM. We change the number of terms from 1 to 8. As expected, with more numbers of terms in the series, more accurate the results are. The same conclusion can be founded in [Reis(2013)], where it was studied the influence of number of terms of the Taylor series in the expansion of fundamental solution for anisotropic problems. With the increasing of number of terms, more accurate the results were found. Another feature is the number of terms used to

obtain the same accuray on matrices H and G. Matrix $[G]$ converges more quickly than matrix $[H]$. One reason for this features is the presence of strong singularity in the fundamental solution $[T_{ij}(\mathbf{z}, \mathbf{z_o})]$. By the end, these results show that the operations of the FMM for anisotropic materials have been completely obtained, being this formualtion one option to solve problems that BEM is inefficient due to large number of unknowns. For fundamental solutions with the presence of singularity, it is necessary to use more numbers of terms on th Taylor series, to preserve the accuracy of the problem.

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