REVISTA INTERDISCIPLINAR DE PESQUISA EM ENGENHARIA

CILAMCE 2016

XXXVII IBERIAN LATIN AMERICAN CONGRESS
ON COMPUTATIONAL METHODS IN ENGINEERING
BRASÍLIA - DF - BRAZIL

ERRORS IN QUANTITIES OF INTEREST IN THE LAMINATED PLATE BENDING PROBLEM USING HIERARCHIC SETS OF BASIS FUNCTIONS IN GFEM

Paulo de Tarso R Mendonça(*)

mendonca@grante.ufsc.br

(*) Federal University of Santa Catarina, Departament of Mecanical Engeneering,

Florianópolis, SC - BRAZIL, CP 476, CEP 88.040-900.

Ludovic Chamoin (**),

chamoin@lmt.ens-cachan.fr

(**) LMT-Cachan, secteur Structures & Systèmes (ENS Cachan/CNRS/Univ.Paris 6)

61 av. du Président Wilson, F-94230 Cachan, FRANCE.

Clovis S. de Barcellos (*)

clovis.barcellos@gmail.com

Abstract. *A formulation for error estimation is developed for the bending problem of composite laminated plates based on the Mindlin-Reissner kinematic model discritized by the Generalized Finite Element Method (GFEM). The error estimation process starts with an upper bound in energy norm, which is obtained following the basic CRE (Constitutive Relation Error) framework of the Ladevèze formulation, that is, the estimate is obtained from a statically admissible stress field computed at element level in a Neumann problem where the element boundary forces are equilibrated. The authors have previously shown that an accurate description of the in plane stresses in a laminate is essential to obtain an accurate approximation to the transverse shear stresses at the layers interfaces. Since important failure modes in laminated composite plates, like the delamination, are linked to the transverse stresses, it is essential to develop both, accurate post-processing procedures to compute improved transverse stresses, and also estimate techniques for the discretization errors. The first condition is adequately satisfied by GFEM. Therefore, the aim of the present work is to extend the general CRE technology to develop*

formulations to estimation of errors in Quantity of Interest (QI) identified preferably with the stress field in the laminated plate problem. One of the steps necessary in the CRE procedure is the computation of and admissible stress field in each element, in a Neumann problem where the boundary forces have been previously equilibrated. For a GFEM basis with high order enrichment, adequate procedures have to be sought. Here we use one single higher order finite element, based on displacement FEM, to obtain an approximation to the equilibrated field. The formulation is implemented for arbitrary degree of the basis, which allows an arbitrarily close approximation to the equilibrium condition. The sharpness of the QI's error bounds is increased with the accuracy of the primal and dual global energy norm of errors. In the present work we investigate the effectiveness of a local GFEM p-enrichment as a tool to improve the approximability of the model in capturing the local gradients which characterizes response of the dual loading. The GFEM p-enrichment is implemented in a simple and straightforward way, as opposed to some other possible forms of enrichment, e.g. local h-refinement or a sub-domain approach. Numerical tests are performed to asses the effect of the different parameters in the modeling over the errors in the quantities of interest.

Keywords: *Strict error bounds estimation, Generalized Finite Element Method, Admissible stress field, Element Equilibrated Technique, Laminated plate Mindlin model.*

1 INTRODUCTION

One of the goals of the Finite Element Method (FEM), or its generalized form (GFEM), in the structural mechanics consists in providing relieble and accurate estimates to be used in analysis and design in the different branches of industry, for example the automobilistic industry, aeronautics, naval, power plants, etc. In the first years of the FEM development, the community was satisfied of merely being able to obtain a numerical approximation of the solution for the problem at hand, but soon after the 1980's, investigations were started aimed to estimate also bounds for the errors commited in the numerical models. More recently, several industrial fields have already made mandatory the execution of an error estimate to every numerical model. Coarse energy estimates have already become available in most large commercial FEM codes.

Among the many techniques and procedures developed to estimate errors in FEM simulations, we can consider, possibly, only a few large families of methods: those based on equilibrated residuals, (Oden & Prudhomme, 2002), the *patch-based flux-free* method, (Parés et al., 2006; Barros et al., 2013), and those based on Constitutive Relation Error,(Ladevèze & Pelle, 2005), hypercircle, (Prager & Synge, 1947), besides many other variations and combinations. These forms have been, since their origin, based on consistent mathematic analysis. In parallel, there are the methods derived from the ZZ method (Zienkiewicz & Zhu, 1992), associated with smoothing of stresses obtained in super convergent points. These methods are more adequate to the post processing of low order finite elements and, only recently, became object of improvements (Díez et al., 2006).

The Constitutive Relation Error method was first proposed by Ladevèze in 1975, initially for the post-processing of FEM results in linear problems, and have been developed further for dynamic and material non linear problems (Ladevèze & Rougeot, 1997; Pelle, 1995; Ladevèze & Chamoin, 2010) for example. Apparently this was the first general purpose method to provide

a strict upper bound for the FEM error, although in 1947 Prager had already developed a method based on hypercircle to give error estimates (Prager & Synge, 1947).

In laminated orthotropic plates it is known (Mendonça et al., 2011) that the transverse shear stresses are poorly estimated by the low order FEM. The extraction method based on transverse stress integration only provide acceptable transverse estimate if the in plane stresses are computed from a FE basis of high order. The GFEM provides a straightforward and simple way to generate p-enrichment in a coarse mesh. Therefore, the GFEM p-enrichment becomes in important tool to generate high order approximation over the plate surface, which provides improved estimates for the transverse shear stresses.

Some research have been done in the development of error estimators for homogeneous isotropic plate FEM models. Benoit in 1999 obtained error bounds in energy norm for the Kirchhoff kinematic model. For the error estimates for Mindlin model, it can be noticed the work of Carstensen in 2006. Da Veiga in 2013 performed error analysis for the MITC plate elements.

The objectives of the present paper are the following: (a) to formulate the global energy error estimate for the laminated plate bending model, based on the Constitutive Relation Error formulation, adapted to the GFEM technique; (b) to formulate the dual problem for the error in quantity of interest. In both steps, it is sought to investigate the effects of p-enrichment in several sub-steps involved, which are, for example, the enrichment of the GFEM basis, of the 1-D polynomial basis used to represent the equilibrated tractions along each element edges, the effect of the basis degree used in the solution of the finite element model for the admissible stress field problem at elements level.

The overall main goal is investigation of the effectiveness of local GFEM *p*-enrichment as a tool to improve the approximability of the model in capturing the local gradients which characterizes response of the dual loading. The GFEM *p*-enrichment is implemented in a simple and straightforward way, as opposed to some other possible forms of enrichment, e.g. local *h*-refinement or a sub-domain approach.

Besides the evaluation of the *p*-enrichment in the dual problem, other evaluations are also performed. The sensibility of the Constitutive Relation Error (CRE) method for a posteriori error estimate in certain types of Quantities of Interest was also evaluated with respect to the degrees polynomial reproducibility used in each of the hierarchic sets of basis functions used in the different stages of the method. The basis degrees tested are the following: (1) The degree d of the FEM basis of the triangle Lagrange functions used for the equilibrium problem in each isolated element problem; (2) The degree e of the 1D polynomial function used on the elements edges to represent the equilibrated resultant forces. All tests were performed with QI's defined as moments of a stress components over a given element. As a novelty, these moments are not only average usually, but they are defined from polynomial weight functions of arbitrary degree g. The tests were done with weights of degrees zero and one.

2 REFERENCE PROBLEM: THE LAMINATED PLATE BENDING

Let us consider the general conditions of a static linear problem of a anisotropic laminated plate occupying a volume $V = \Omega \times I \in R^3 \times R$, where Ω represents the reference (mean) surface, with cartesian coordinates $\mathbf{x} = (x, y)$. The boundary of the reference surface is designated as Γ. I is the domain along the the transverse direction, defined as $I = \{z$, such that

$z \in [-H/2, H/2]\}$, where H is the laminate thickness. The boundaries of the plate are defined by the surfaces $z = \pm H/2$ and are designated as S^+ and S^-, and the lateral surface L, defined by $(\mathbf{x}, z)$ such that $\mathbf{x} \in \Omega$ and $z \in I$.

The plate is subjected to volumetric body load $\mathbf{b} = \mathbf{b}(\mathbf{x}, z)$, surface distributed load $\mathbf{q} = \mathbf{q}(\mathbf{x})$ applied over the upper surface S^+, and lateral surface loads $\mathbf{f}$ applied at the part of the lateral border L designated as L_f. Here we restrict the distributed load to have only the transverse component, that is, in cartesian coordinates, $\mathbf{q} = \{0, 0, q_z)$. The rest of the boundary conditions are: (a) free surface at S^-; (b) Dirichlet boundary conditions $\mathbf{u} = \bar{\mathbf{u}}$ at the part of the lateral surface designated as L_u. The vectors $\mathbf{b}$, $\mathbf{f}$, $\mathbf{q}$ and $\mathbf{u} \in \mathbf{R}^3$.

The kinematic relations for the First Order Shear Deformation Theory (Mindlin's Model) can be represented as

$$u_x(\mathbf{x}, z) = u_x^0(\mathbf{x}) + z\psi_x(\mathbf{x}), \quad u_y(\mathbf{x}, z) = u_y^0(\mathbf{x}) + z\psi_y(\mathbf{x}), \quad u_z(\mathbf{x}, z) = w(\mathbf{x}). \tag{1}$$

Where u_x^0 and u_y^0 are generalized in plane displacement components of a point in the reference surface, ψ_x, ψ_y are warp functions and w is the transverse displacement of the reference surface. The tests described in this paper can be done more clearly, without loss of generality, restricting the model to simple bending, that is, transverse loading on symmetric laminate. Therefore, the **finite element solution of the equilibrium bending problem** is: *given* q_z, $\bar{\chi}_m$ *and* $\overline{Q}_n$, *obtain* F_h *that satisfies*

$$\int_\Omega F_h \cdot \Sigma_h(\mathbf{u}_h^*)\, d\Omega - \int_\Omega \mathbf{q} \cdot\ \mathbf{u}_h^*\, d\Omega - \int_{\Gamma_f} \bar{\mathbf{t}} \cdot \mathbf{u}_h^*\, d\Gamma = 0, \textit{for any } \mathbf{u}_h^* \in Var_h, \tag{2}$$

where the symbols were adapted from the general problem to the bending case as

$$\Sigma_h = \left\{ \begin{array}{c} \kappa \\ \gamma_c \end{array} \right\}, \quad F_h = \left\{ \begin{array}{c} \mathbf{M}_h \\ \mathbf{Q}_h \end{array} \right\}, \quad \mathbf{M}_h = \left\{ \begin{array}{c} M_x \\ M_y \\ M_{xy} \end{array} \right\}_h, \quad \mathbf{Q}_h = \left\{ \begin{array}{c} Q_y \\ Q_x \end{array} \right\}_h,$$

$$\mathbf{q} = \left\{ \begin{array}{c} q_z \\ 0 \\ 0 \end{array} \right\}, \quad \bar{\mathbf{t}} = \left\{ \begin{array}{c} \overline{Q}_n \\ \overline{X}_m \\ \overline{Y}_m \end{array} \right\} \quad \text{and } \mathbf{u}_h = \left\{ \begin{array}{c} w \\ \psi_x \\ \psi_y \end{array} \right\}_h. \tag{3}$$

The constitutive relation for the laminate, when restricted to bending, will be represented in compact form as $F = \mathbf{C}\Sigma$ where $\mathbf{C}$ is the constitutive matrix for the laminate at the reference surface coordinate $\mathbf{x}$.

3 GFEM DISCRETIZATION

Let us consider an usual FEM regular mesh of elements of domain Ω_e defined such that $\Omega_e \subset \Omega$ and $\cup\Omega_e = \Omega$. The mesh is defined by a set of nodes $n = 1, ..., nno$ of coordinates

$\mathbf{x}_n = (x_n, y_n)$, and define the patch P_n as the union of all elements that share node n, whose domain is Ω_{Pn} Let us define a set of partition of unity (PoU) functions $\phi(\mathbf{x}) = \{\phi_1, \phi_2, \cdots, \phi_n, \cdots, \phi_{nno}\}$, where ϕ_n is associated with patch P_n such that $\phi_n(x_n) = 1$ and it has compact support on Ω_{Pn}.

The GFEM general foundation consists in obtaining a set of approximation functions by enriching the PoU set by adequate functions. In many cases these are singular or high gradient regular functions defined around a localized region on the domain. However, for the goals of this paper, it is more adequate to use the most traditional setting of GFEM, based on enrichment given by a simple complete set of monomials defined in global coordinates. In this way the tests will give results more clearly in a controlled set of problems. Therefore, we use the set of nodal monomials $\mathcal{E}_n = \{1, \bar{x}, \bar{y}, \bar{x}^2, \bar{x}\bar{y}, \bar{y}^2, \bar{x}^3, \bar{x}^2\bar{y}...\}$, where $\bar{x}$ and $\bar{y}$ are normalized coordinates, arbitrarily defined as $\bar{x} = (x - x_n)/L_{xn}$ and $\bar{y} = (y - y_n)/L_{yn}$ and L_{xn} and L_{yn} are characteristic lengths of the patch P_n. The set of all nodal monomials is $\mathcal{E} = \{\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n, ..., \mathcal{E}_{nno}\}$. A set of enriched approximation functions is generated as

$$N(\mathbf{x}) = \phi(\mathbf{x})\mathcal{E} = \{\phi_1\mathcal{E}_1,\ \phi_2\mathcal{E}_2,\ \cdots, \phi_n\mathcal{E}_n,\ \cdots, \phi_{nno}\mathcal{E}_{nno}\}. \tag{4}$$

Non homogeneous enrichment can be easily defined, using a different set $\mathcal{E}_n$ for each patch. The set $N(\mathbf{x})$ is used to discretize the generalized displacement components $\mathbf{u}_h$, for example, $w(\mathbf{x}) = N(\mathbf{x})\mathbf{W}$, where $\mathbf{W}$ is a convenient set of nodal coefficients of the approximation. After that, the usual displacement based FEM formulation follows naturally.

It is possible to prove that the set of approximation functions $N(\mathbf{x})$ is capable of approximating precisely any complete polynomial equation up to degree p, over a patch P_n, if the patches of all the nodes q in P_n are enriched with complete sets $\mathcal{E}_q$ of monomials of degree p. Throughout the text, we will refer to a set of *GFEM basis degree* p.

The usual form of the PoU used is the $C^0(\Omega)$ shape functions used in FEM. Most commonly, the linear piecewise functions. This gives origin to what we call C^0-GFEM, in oposition to the smooth versions, called C^k-GFEM, (Mendonça et al., 2011; 2013) where the PoU functions are $C^k(\Omega)$ continuous, where k is a positive integer. The formulation for error estimate described here is adequate for both GFEM forms.

4 CONSTITUTIVE RELATION ERROR ESTIMATOR

Let us consider three pairs of solution:

1. The pair $(\mathbf{u}_{ex}, \digamma_{ex})$, which is the exact solution of the complete set of conditions: the equilibrium condition (2), the constitutive equations and the kinematic relations eq.(1) and $\mathbf{u} \in Kim$;
2. The approximate FE solution pair $(\mathbf{u}_h, \digamma_h)$, with $\mathbf{u}_h \in Kim_h$ and $\digamma_h$ such that $\mathbf{M}_h = \mathbf{D}\kappa(\mathbf{u}_h)$ and $\mathbf{Q}_h = \mathbf{E}\gamma^c(\mathbf{u}_h)$. Thus, the discretization error is defined as $e_u = \mathbf{u}_{ex} - \mathbf{u}_h$;
3. An admissible solution $(\hat{\mathbf{u}}_h, \hat{\digamma}_h)$, that can be derived from $(\mathbf{u}_h, \digamma_h)$. It is admissible in the sense that $\hat{\mathbf{u}}_h \in Kim_h$, (it is kinematically admissible) and $\hat{\digamma}_h$ satisfies (2) (it is statically admissible).

It is considered two measures of the discretization error: a global error, based on the energy norm and a local one defined with respect to a given quantity of interest. The global error is defined as

$$E_{gl} = \|e_u\|_{Kim,\Omega} = \|\digamma_{ex} - \digamma_h\|_{S,\Omega} \tag{5}$$

where the norms are defined as

$$\|\bullet\|_{Kim,\Omega} = \left(\int_\Omega \mathbf{C}\Sigma(\bullet) : \Sigma(\bullet)\, d\Omega\right)^{1/2} \quad \text{and} \quad \|\bullet\|_{S,\Omega} = \left(\int_\Omega \bullet : \mathbf{C}^{-1} \bullet\, d\Omega\right)^{1/2}. \tag{6}$$

The Constitutive Relation Method (CRE) adopted in this work uses an admissible solution to derive a strict, computable upper bound of the global error E_{gl}. First, the quantity $E_{CRE}(\mathbf{v},\digamma)$ is defined as

$$E_{CRE}(\mathbf{v},\digamma) \equiv \left(\frac{1}{2}\,\|\digamma - \mathbf{C\Sigma}(\mathbf{v})\|^2_{S,\Omega}\right)^{1/2} \quad \text{for } \forall(\mathbf{v},\digamma). \tag{7}$$

It is proved in Ladevèze & Pelle (2005) that

$$\|\mathbf{u}_{ex} - \hat{\mathbf{u}}_h\|^2_{Kin,\Omega} + \left\|\digamma_{ex} - \hat{\digamma}_h\right\|^2_{S,\Omega} = 2E^2_{CRE}(\hat{\mathbf{u}}_h,\hat{\digamma}_h), \tag{8}$$

such that E_{CRE} produces an upper bound as $E_{gl} \leq \sqrt{2}E_{CRE}(\hat{\mathbf{u}}_h,\hat{\digamma}_h)$. The computation of $\hat{\digamma}_h$ can be performed by different techniques, but probably the most versatile are the element equilibration technique (Ladeveze & Pelle, 2005) and the flux-free technique (Parés et al., 2006; Díez et al., 2006). For the admissible displacement, usually one simply takes $\hat{\mathbf{u}}_h = \mathbf{u}_h$.

5 UPPER AND LOWER BOUNDS ON ERROR OF QI - from CRE

Consider that the **quantity of interest** (QI) is a linear functional $\mathcal{L}(\mathbf{u})$ of the displacement field, defined in the form

$$Q(\mathbf{u}) = \mathcal{L}(\mathbf{u}) \equiv \int_\Omega \left(\tilde{\sigma}_\Sigma : \varepsilon(\mathbf{u}) + \tilde{\digamma}_\Sigma : \Sigma(\mathbf{u}) + \tilde{\mathbf{f}}_\Sigma \cdot \mathbf{u}\right) d\Omega, \tag{9}$$

where $\tilde{\sigma}_\Sigma$, $\tilde{\digamma}_\Sigma$ and $\tilde{\mathbf{f}}_\Sigma$ are operators (called *extractors*) with dimensions of stresses, resultant stresses and force respectively, arbitrarily chosen to provide the quantity one has interest, and can be given in explicit or implicit form. $\tilde{\digamma}_\Sigma$ is adequate to identify quantities of interest of generalized deformation of the reference surface, that is, $\{\kappa_x, \kappa_y, \kappa_{xy}, \gamma_{yz}, \gamma_{xz}\}^T$, $\tilde{\sigma}_\Sigma$ is adequate to identify deformation components, that is, $\{\varepsilon_x, \varepsilon_y, \gamma_{xy}, \gamma_{yz}, \gamma_{xz}\}^T$ and $\tilde{\mathbf{f}}_\Sigma$ identifies generalized displacements of a point in the reference surface, that is, $\{w, \psi_x, \psi_y\}^T$. In the next section they

will be adapted to identify the more useful quantities of interest of stresses, $\{\sigma_x,\ \sigma_y,\ \tau_{xy},\ \tau_{yz},\ \tau_{xz}\}^T$ at a given point $(\mathbf{x}, z)$ in the volume, resultant stresses $\{M_x,\ M_y,\ M_{xy},\ Q_y,\ Q_x\}^T$ and distributed forces $\{q,\ m_x,\ m_y\}^T$, respectively.

Next, the following **adjoint problem** is defined, which uses $\mathcal{L}(\mathbf{u})$ as a weak form loading: *find the displacement-resultant stress pair* $(\tilde{\mathbf{u}},\tilde{F})$ *which satisfy*

$$\begin{aligned}
&\text{Kinematic conditions:} && \tilde{\mathbf{u}} \in Kin \quad \text{and } \tilde{\mathbf{u}}(\mathbf{x}) = \mathbf{0}, && \text{for } \forall \mathbf{x} \in \mathbf{\Gamma}_u,\\
&\text{Equilibrium condition:} && \int_\Omega \tilde{F} \cdot \Sigma(\mathbf{u}^*) d\Omega = \mathcal{L}(\mathbf{u}^*), && \text{for } \forall \mathbf{u}^* \in \mathcal{U}_0,\\
&\text{Constitutive relation:} && \tilde{F}(\mathbf{x}) = \mathbf{C}\Sigma(\tilde{\mathbf{u}}(\mathbf{x})) && \text{for } \forall \mathbf{x} \in \Omega.
\end{aligned} \tag{10}$$

The structure of the adjoint problem is the same as the reference problem, but with a different loading. At this point, one has two problems, with two approximate solutions, and it is necessary to estimate the discretization errors on each approximation. Using the CRE methodology, one obtains FE approximations for each problem, which does not satisfy local equilibrium, and from then, one generates admissible solutions, which satisfyes all conditions except the constitutive relation. Diagrammatically, one has:

$$\begin{aligned}
\text{Reference problem:} \quad & (\mathbf{u}_h,F_h) \text{ (FE approx.)} \longrightarrow (\hat{\mathbf{u}}_h,\hat{F}_h) \text{ (admissible)},\\
\text{Adjoint problem:} \quad & (\tilde{\mathbf{u}}_h,\tilde{F}_h) \text{ (FE approx.)} \longrightarrow (\widehat{\tilde{\mathbf{u}}}_h,\widehat{\tilde{F}}_h) \text{ (admissible)}.
\end{aligned} \tag{11}$$

The admissible stresses are computed from the **prolongation condition** which, for the adjoint problem is the left equality of:

$$\int_\Omega \widehat{\tilde{F}}_h \cdot \mathbf{\Sigma}(\mathbf{u}^*) d\Omega = \int_\Omega \tilde{F}_h \cdot \mathbf{\Sigma}(\mathbf{u}^*) d\Omega = \mathcal{L}(\mathbf{u}^*). \tag{12}$$

With the two approximate admissible solutions of (11), the **Theorem 1** below is stated, as a modification of a theorem in ref. (Ladevèze & Chamoin, 2010) that was enunciated in the frame of linear elasticity and here it is adapted to relate quantities of interest of stresses and resultant stresses which are characteristic of the laminated plate model. Thus, the bounds for the **discretization error in quantity of interest** are estimated from the following theorem.

Theorem - *Consider Q_{ex} the unknown exact value of the QI and Q_h the computed FE approximation of it. One has the following inquality*:

$$|Q_{ex} - Q_h - Q_{hh}| \leq \underbrace{E_{CRE}(\hat{\mathbf{u}}_h,\hat{F}_h)}_{E}\ \underbrace{E_{CRE}\left(\widehat{\tilde{\mathbf{u}}}_h,\widehat{\tilde{F}}_h\right)}_{\tilde{E}}, \quad \textit{where} \tag{13}$$

$$Q_{hh} \equiv \int_\Omega \left(\hat{F}_h - \mathbf{C}\mathbf{\Sigma}(\hat{\mathbf{u}}_h)\right)\cdot\mathbf{C}^{-1}\widehat{\tilde{F}}_h^m\, d\Omega + \mathcal{L}(\hat{\mathbf{u}}_h - \mathbf{u}_h) \quad \text{and} \quad \widehat{\tilde{F}}_h^m = \frac{1}{2}\left(\widehat{\tilde{F}}_h + \mathbf{C}\mathbf{\Sigma}(\widehat{\tilde{\mathbf{u}}}_h)\right). \tag{14}$$

———

The inequality (13) can be written as

$$
\begin{aligned}
|Q_{ex} - Q_h - Q_{hh}| &\leq E\ \tilde{E}, \\
-E\ \tilde{E} \leq Q_{ex} - Q_h - Q_{hh} &\leq E\ \tilde{E}, \\
\boxed{\underbrace{Q_{hh} - E\ \tilde{E}}_{\eta_Q^L} \leq Q_{ex} - Q_h \leq \underbrace{E\ \tilde{E} + Q_{hh}}_{\eta_Q^U}} &
\end{aligned}
\tag{15}
$$

such that one obtains the desired bounds η_Q^L and η_Q^U for the **discretization error** of the quantity of interest. Also, one can obtain bounds for the QI itself: $\eta_Q^L + Q_h \leq Q_{ex} \leq \eta_Q^U + Q_h$.

From (15), the average of the error in the QI is

$$
(Q_{ex} - Q_h)_m \equiv \frac{1}{2}\left(\eta_Q^L + \eta_Q^U\right) = Q_{hh}. \tag{16}
$$

This shows explicitly the role of Q_{hh} as the center of the range between the lower and upper bounds of the error, which is given by the product $E\ \tilde{E}$. This term, $E\ \tilde{E}$, is the only one present in the previous error estimate formulations prior to the work Ladevèze & Chamoin in 2010, which included the term Q_{hh}. Part of the goals of the present work is to investigate the efficiency of this estimation.

6 SOME DETAILS ON THE USE OF THE EXTRACTORS

Consider that we want to have an error estimate of a given resultant stress component, $F_1(\mathbf{x})$, **at all** $\mathbf{x} \in \Omega_\Sigma$, where $\Omega_\Sigma \subseteq \Omega$ is an arbitrary region of the domain, for example, a chosen finite element of the mesh or some smaller region in the element or still a region encompassing a group of elements of partes of then. The definition in (9) makes use of an integral over the domain. This formal structure can be adapted in the following way. Let us consider, for the sake of clarity of description, only two of the three extractors in in (9), the in plane components of $\tilde{\sigma}_\Sigma$ and $\tilde{\mathbf{f}}_\Sigma$, and decompose $\tilde{\sigma}_\Sigma$ with help of the material property matrix

$$
Q(\mathbf{u}) = \mathcal{L}(\mathbf{u}) \equiv \int_\Omega \left(\tilde{\varepsilon}_\Sigma \cdot \sigma(\mathbf{u}) + \tilde{\mathbf{f}}_\Sigma \cdot \mathbf{u}\right) d\Omega. \tag{17}
$$

$\sigma = \{\sigma_x, \sigma_x, \tau_{xy}\}^T$ are the in plane stresses at a point $(\mathbf{x}, z)$ in the volume and $\mathbf{S}^k$ is the material matrix of the layer k which contains the coordinate z. For bending, $\varepsilon = z\kappa$. Let us consider a set of continuous basis functions defined over $\mathbf{\Omega}_\Sigma$, represented as $\mathbf{N}(\mathbf{x}) = \{f_1, f_2, \cdots, f_k, \cdots, f_n\}$, where $f_k(\mathbf{x})$ has compact support on Ω_Σ. If we are interested in the error estimate of a given stress component of σ, that is, σ_p, $p = 1, 2, 3$, we define a set of n deformation extractor vectors $\tilde{\varepsilon}^p_{\Sigma g}$, $g = 1, 2, \cdots, n$, such that its only non zero entry is $f_k(\mathbf{x})$, at position p. For example, for σ_x, $p = 1$, the set becomes

$$\begin{aligned}
&\tilde{\varepsilon}^1_{\Sigma 1} = \{f_1(\mathbf{x}), 0, 0\}^T, \quad \tilde{\varepsilon}^1_{\Sigma 2} = \{f_2(\mathbf{x}), 0, 0\}^T, \quad \tilde{\varepsilon}^1_{\Sigma g} = \{f_k(\mathbf{x}), 0, 0\}^T, \\
&\quad \vdots \\
&\tilde{\varepsilon}^1_{\Sigma n} = \{f_n(\mathbf{x}), 0, 0\}^T, \text{ etc, if } \mathbf{x} = (x, y) \in \mathbf{\Omega}_\Sigma \text{ and } f_g = 0 \text{ if } (x, y) \notin \mathbf{\Omega}_\Sigma.
\end{aligned} \tag{18}$$

The functions $f_k(\mathbf{x})$ must form a complete set, whether polynomial, trigonometric or other. The set $\tilde{\varepsilon}^p_{\Sigma g}$ produce n moments of the QI:

$$\begin{aligned}
\mathcal{L}_g(\mathbf{u}) &= \int_{\Omega_\Sigma} \tilde{\varepsilon}^p_{\Sigma g} \cdot \mathbf{S}^k \varepsilon(\mathbf{u}) d\Omega_\Sigma, \\
&= \int_{\Omega_\Sigma} \tilde{\varepsilon}^p_{\Sigma g} \cdot \sigma(\mathbf{u}) d\Omega_\Sigma, \quad g = 1, 2, \cdots, n.
\end{aligned} \tag{19}$$

It is clear that $\mathcal{L}_g(\mathbf{u})$ is the QI of the moments of $\sigma_x(\mathbf{x})$ with respect to the function f_g. If one computes FE approximations for all of these moments, and the respective bounds of error accordingly to (15), one has, in simplified notation, for a given moment k,

$$L^g_Q \leq \int_{\Omega_\Sigma} f_g \, \Delta\sigma_x \, d\Omega_\Sigma \leq U^g_Q. \tag{20}$$

$\Delta\sigma_x(\mathbf{x}) = (\sigma_{xex} - \sigma_{xh})$ is the unknown pointwise error in the stress component. It is possible to arbitrate another set of basis functions $\bar{\mathbf{N}}(\mathbf{x}) = \{\bar{f}_1, \bar{f}_2, \cdots, \bar{f}_g, \cdots, \bar{f}_m\}$ to discretize the error in the stress component, such that, for any $\mathbf{x} \in \Omega_\Sigma$,

$$\begin{aligned}
\Delta\sigma_x(\mathbf{x}) &= \sigma_{xex} - \sigma_{xh} \simeq \bar{\mathbf{N}}(\mathbf{x}) \Delta\mathbf{S}, \\
&= \bar{f}_l(\mathbf{x}) \Delta S_l, \quad l = 1, 2, ..., n,
\end{aligned} \tag{21}$$

where $\bar{\mathbf{N}}(x, y)$ is a set of m basis functions on Ω_Σ and $\Delta\mathbf{S}$ is the correspondent set of unknown coefficients.

Substituting this approximation in (20) and performing the integration, one obtains

$$\begin{aligned}
&L^g_Q \leq \left[\int_{\Omega_\Sigma} f_g \, \bar{\mathbf{N}} \, d\Omega_\Sigma \right] \Delta\mathbf{S} \leq U^g_Q, \qquad \text{for } g = 1, 2, ..., n, \\
&L^g_Q \leq \underbrace{\left[\int_{\Omega_\Sigma} f_g \, \bar{f}_l \, d\Omega_\Sigma \right]}_{G_{gl}} \Delta S_l \leq U^g_Q, \quad l = 1, ...n, \\
&L^g_Q \leq G_{gl} \, \Delta S_l \leq U^g_Q, \\
&\mathbf{L}_Q \leq \mathbf{G} \, \Delta\mathbf{S} \leq \mathbf{U}_Q, \quad \text{(in vector form)}
\end{aligned} \tag{22}$$

$\mathbf{G}$ is a $n \times m$ matrix, whose components are

$$G_{gl} = \int_{\Omega_\Sigma} f_g \; \bar{f}_l \; d\Omega_\Sigma. \tag{23}$$

(If the set of functions chosen for f_g and $\bar{f}_l$ are the same and linearly independent, $\mathbf{G}$ will be square, symmetric and positive definite.) Equation (22) defines two algebraic problems $\mathbf{G}\ \Delta\mathbf{S} = \mathbf{L}_Q$ and $\mathbf{G}\ \Delta\mathbf{S} = \mathbf{U}_Q$. If $n \geq m$, the solution is unique and (22) becomes

$$\mathbf{p} \leq \ \Delta\mathbf{S} \leq \mathbf{q}, \tag{24}$$

where $\mathbf{p} = \mathbf{G}^{-1}\mathbf{L}_Q$ and $\mathbf{q} = \mathbf{G}^{-1}\mathbf{U}_Q$ are coefficient vectors of the errors bounds, associated with the basis $\bar{\mathbf{N}}$.

These coefficients are next used to combine the terms in the basis $\bar{\mathbf{N}}$ to generate the pointwise error of $\Delta\sigma_1$:

$$\begin{aligned} \bar{\mathbf{N}}\mathbf{p} \leq \bar{\mathbf{N}}\ \Delta\mathbf{S} \leq \bar{\mathbf{N}}\mathbf{q}, \\ \bar{f}_l p_l \leq \bar{f}_l\ \Delta S_l \leq \bar{N}_l q_l, \\ L_Q(\mathbf{x}) \leq \Delta\sigma_x(\mathbf{x}) \leq U_Q(\mathbf{x}). \end{aligned} \tag{25}$$

In this way, we have obtained $L_Q(\mathbf{x})$ and $U_Q(\mathbf{x})$, the bounds for the **pointwise error** of the component σ_x at the point $\mathbf{x} \in \mathbf{\Omega}_\Sigma$. **It should be noted** that, even if the bounds for the error in each moment of QI are strict, the same cannot be said from the bounds in (25), due to the approximation done on the error of the stress, eq. (21), whose accuracy is dependent of the size of basis $\bar{\mathbf{N}}$. In all tests we used the triangular Lagrangian basis functions for both $\bar{\mathbf{N}}$ and $\mathbf{N}$. It is possible to obtain bounds for the pointwise error of a stress component, at any position in an arbitrary region Ω_Σ. The region is taken here to be an entire arbitrary element in the mesh.

7 ASPECTS ON THE EQUILIBRATED ELEMENT TECHNIQUE

The equilibrated element technique for the FEM is well established after some decades, Oden & Prudhomme (2002) and Ladeveze & Pelle (2005), but here we summarize some of its main steps in order to be able to describe some particularities of its application to the GFEM when it is implemented with hierarchic polynomial enrichment.

Given the FEM or GFEM results, the general steps for construction of an equilibrated resultant force field, $\hat{F}_h(\mathbf{x})$, $\mathbf{x} \in \mathbf{\Omega}$, are the following.

1. For each element E, construction of a resultant forces field on the element edges which satisfies interelement continuity with neighboring elements, and is it equilibrated with the loading and internal forces in the element;
2. For each element E, construction of an statically admissible resultant force field $\hat{F}_h(\mathbf{x})$, $\mathbf{x} \in \Omega_E$.

The core of the technique is the *prolongation condition* which, in the laminated plate problem can be expressed in the following way. Given an approximate solution $\mathbf{M}_h$, $\mathbf{Q}_h$ obtained by the displacement based finite element method, with an approximation basis functions set $\varphi_{ij}(\mathbf{x})$ where $i = 1, 2, ...$number of approximation functions in direction j, for $j = 1, 2, 3$ for w, θ_x, θ_y respectively. One seeks an equilibrated field $\hat{F}_h = \{\hat{\mathbf{M}}_h, \hat{\mathbf{Q}}_\mathbf{h}\}$, as a prolongation of the finite element solution in the sense: *For any approximation function $\varphi_{ij}(\mathbf{x})$ of the finite element model, and for any element E of the mesh*, it is imposed that

$$\int_{\Omega_E} \hat{F}_h \cdot \Sigma(\mathbf{u}^{*j}(\varphi_{ik}))\, d\mathbf{\Omega}_E = \int_{\Omega_E} F_h \cdot \Sigma(\mathbf{u}^{*j}(\varphi_{ik}))\, d\mathbf{\Omega}_E, \tag{26}$$

where j indicates the components of the triad of generalized bending displacements in $\hat{\mathbf{u}}$: $\mathbf{u}^* = \left\{w^*,\ w^*,\ \theta_y^*\right\}^T$. In this way, the equilibrated forces in the element are set to have the same deformation energy in bending as the the FEM approximation, due to the linearity of (26) with respect to φ_{ij}. Observe that $\varphi_{i1} = \{\varphi_i;0;0\}^T$ or $\varphi_{i2} = \{0;\varphi_i;0\}^T$ or $\varphi_{i3} = \{0;0;\varphi_i\}^T$.

Then we obtain an expression for the boundary tractions

$$\int_{\partial\Omega_E} \hat{\mathbf{T}} \cdot \mathbf{u}^{*j}(\varphi_{ik})\ d\Gamma = \underbrace{\int_{\Omega_E} F_h \cdot \Sigma(\mathbf{u}^{*j}(\varphi_{ik}))\, d\mathbf{\Omega}_E - \int_{\Omega_E} \mathbf{q} \cdot\ \mathbf{u}^{*j}(\varphi_{ik})\, d\Omega}_{D_E^j(i,k)}. \tag{27}$$

where, to simplify, we consider only transverse distributed load q_z, such that $\mathbf{q} = \{q_z,\ 0,\ 0\}^T$ and boundary tractions $\mathbf{T}$ on the element are related to the generalized force by

$$\mathbf{T} \equiv \left\{ \begin{array}{c} Q_n \\ X_m \\ Y_m \end{array} \right\}_{\partial E} = \left[\begin{array}{ccccc} & & & n_x & n_y \\ n_x & & n_y & & \\ & n_y & n_x & & \end{array} \right] \left\{ \begin{array}{c} M_x \\ M_y \\ M_{xy} \\ Q_x \\ Q_y \end{array} \right\}. \tag{28}$$

n_x and n_y are the cartesian components of the normal vector on the boundary, outward to the element.

The right hand side of (27) is known and is represented by $D_E^j(i,k)$. Due to this decoupling of $\hat{\mathbf{u}}^j$, eq.(27) generates three algebraic equations, one for each direction j, which produces the values of the corresponding component of the equilibrated boundary traction $\hat{T}_j$ in the element E. Then (27) can be written as

$$\int_{\partial E} \hat{T}_j \varphi_{ik}\, d\Gamma = D_E(i,j,k), \text{ for node } i, \text{ component } j, \text{ element } E. \tag{29}$$

For a triangular element, all functions φ_{ik} are zero on the edge opposite to the node i. Therefore, th integral at the left hand side, on the boundary of the E_n-th ($n = 1, 2, \cdots, N$) element connected to the node i, is defined only by the two edges issuing from node i. The value on each edge $r = 1, 2$ is designated by $b_n^r(i,k)$ (**interface density projection**), that is

$$b_n^r(i,j,k) \equiv \int_{\Gamma^r_{E_n}} \hat{T}_j \varphi_{ik} \, d\Gamma, \tag{30}$$

where $\Gamma^r_{E_n}$, for $r = 1, 2$, are the edges of the element E_n connected to the node i. Also, the right hand side of (29) is

$$D_{E_n}(i,j,k) \equiv \int_{\Omega_E} \boldsymbol{F}_h \cdot \Sigma(\mathbf{u}^{*j}(\varphi_{ik})) \, d\Omega_E - \int_{\Omega_E} \mathbf{q} \cdot \mathbf{u}^{*j}(\varphi_{ik}) \, d\Omega. \tag{31}$$

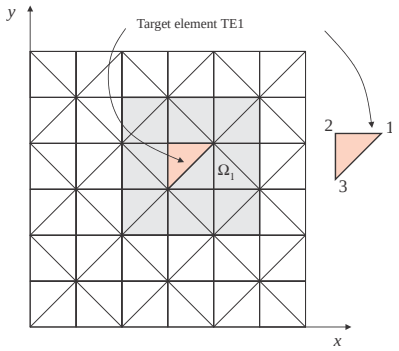

Figure 1: GFEM mesh in the laminate, showing the target element TE1 for QI and the associated surrounding subdomain Ω_1. In the detail, local node numbers of TE1.

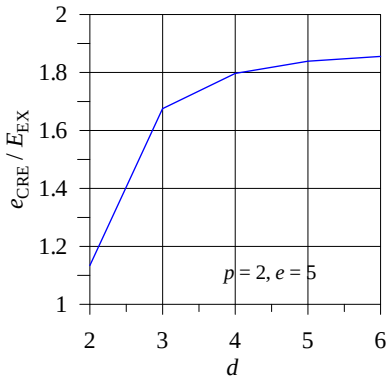

Figure 2: Estimated energy norm of error in primal problem from CRE versus the degree d of the FEM basis in the equilibrium problem. $e_{CRE} = \sqrt{2}E_{CRE}$, and E_{EX} is the exact energy norm of error. $p = 2$, $e = 5$.

8 RESULTS

The test problem chosen is the classical problem of the square laminated plate, simply supported, subjected to a transverse distributed load, with symmetric layer stack. The advantage

of this model is that it possesses complete exact solution for the Mindlin kinematic model, (Dobyns, 1981)**,** for all types of response: displacements, stress components and strain energy. The data are the following: sides $a = b = 200$ mm, total thickness $H = 50$ mm, three equal orthotropic layers oriented at [0°,90°/0°] of equal thicknesses h, and elastic properties $E_1 = 175$ GPa, $E_2 = 7$ GPa, $G_{12} = G_{13} = 3.5$ GPa, $G_{23} = 1.4$ GPa, $\nu_{12} = \nu_{23} = 0.25$, $k = 1.0$, where the subscripts 1 and 2 indicate the principal orthotropic material directions, such that direction 3 coincide with global axis z and 1 with x in layers 1 and 3. The applied load is transverse sinusoidal distributed load given by $q(x, y) = q_0 \sin(\pi x/a) \sin(\pi y/b)$, with $q_0 = 0.001$ N/mm^2.

A single mesh is used, with three noded triangular elements, shown in Figure 1. For the purpose of the tests, the quantity of interest is defined over one single element, denominated target element, TE1, indicated in the figure. The region designated as Ω_1 around TE1 is enriched for the approximate solution of the dual problem. The definition of the QI is made with use of weight functions which are selected as the triangular Lagrangian functions associated with the TE1. The results are shown only for constant and linear weight functions, which are associated with the intrinsic nodes 1,2 and 3 shown in Figure 1 for TE1.

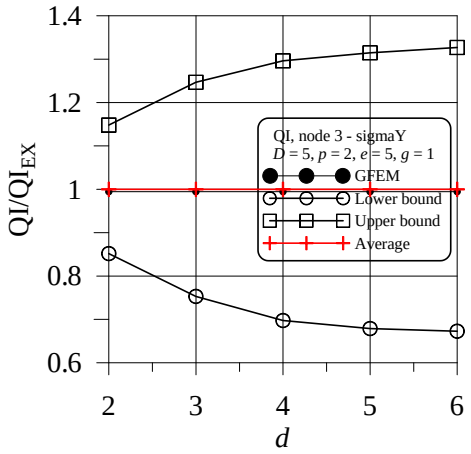

Figure 3: QI for node 3 of the target element TE1, defined with linear weight function, for σ_y, versus the degree d of the FEM basis in the equilibrium problem. $D = 5$, $p = 2$, $e = 5$.

The goal of the present paper is to evaluate the sensibility of the Constitutive Relation Error (CRE) method for a posteriori error estimate in certain types of Quantities of Interest, with respect to the degrees polynomial reproducibility in each of the hierarchic sets of basis functions used in the different stages of the method. These basis degrees are the following:

1. $p = 2, 3, ..., 9$ is the degree of the GFEM basis in the primal global problem. An uniform enrichment is adopted to simplify the observations;
2. $D = 2, 3, 4, 5$ is the degree of the GFEM basis in the dual problem, at the region around the QI element (subdomain Ω_1 indicated in Figure 1). (Here there are results only for QI target element TE1, of Figure 1, not for TE1).
3. $d = 2, ..., 9$ = FEM basis degree (triangle Lagrange functions) for the equilibrium problem in each isolated element problem;
4. $g = 0, 1$ = degree of the set of polynomials used as weight functions in the definition of the QI, the functions $f_k(\mathbf{x})$ in (19).

The first test is the the effectivity index for the primal problem, shown in Figure 2, for e_{CRE}/E_{EX} versus d, where $e_{CRE} = \sqrt{2}E_{CRE}$ is the estimated upper bound of error in energy norm of the primal problem normalized by E_{EX}, the exact GFEM error in energy norm. It was kept fixed the degree $p = 2$ in the GFEM model and degree $e = 5$ in the 1D polynomial for the equilibrated resultant forces on the element edges. It is seem that e_{CRE}/E_{EX} grows asymptotically to abound about 1.8. Small values of d seems to generate, in this specific problem, more accurate approximations, although it is known that only the asymptotic value is a guaranteed upper bound. Different problems show that small values of d produce effectivity indices smaller than one.

Next we start with the evaluations for the QI's. In all results only one entity is chosen, the stress component σ_y integrated over the target element TE1. In some of the results the weight function $f_k(\mathbf{x})$ in (19) is the unity constant, that is, a polynomial of degree $g = 0$, and other in other results $g = 1$ is used, where the polynomial is one of the linear Lagrangean functions associated with one of the three nodes of the element TE1.

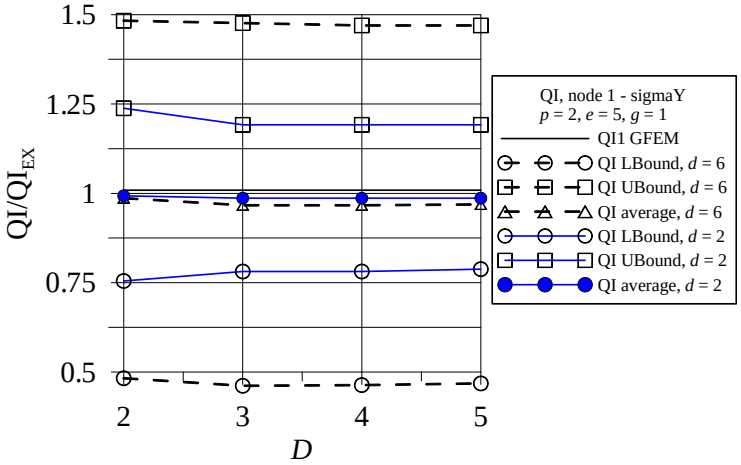

Figure 4: QI of the target element TE1, defined with linear weight function for node 1, for σ_y, versus the degree D of the dual GFEM basis. $p = 2$, $e = 5$.

Figure 3 shows the evolution of the QI defined with linear and constant weight functions, for σ_y, versus the degree d of the FEM basis in the equilibrium problem. The estimates are normalized by QI_{EX}, the exact value of the QI. It is observed and asymptotic behavior of the bounds with growth of d, similar for both types of weight functions, with $g = 0$ or 1. In both cases the range of the bounds with respect to the average is about $\pm 30\,\%$. The curve indicated as "Average" is obtained directly as the average of the upper and lower bound values. Considering (15), the results denoted as QI upper and lower values are computed as

$$\begin{aligned} \text{QI}_L &= Q_h + Q_{hh} - E\ \tilde{E}, \\ \text{QI}_U &= Q_h + Q_{hh} + E\ \tilde{E}, \end{aligned} \tag{32}$$

such that the average is

$$\text{QI}_m \equiv \frac{1}{2}\left(\text{QI}_L + \text{QI}_U\right) = Q_h + Q_{hh}. \tag{33}$$

The results in Figures 3 and the ones that follows show a striking accuracy of the estimation of the QI by the average of the bounds, for all values of d. This is a direct result of the term Q_{hh} of the formulation, which issues from the framework of the hypercircle theorem.

Figure 4 evaluates the effect of D, the degree of the GFEM basis in the region around the target element TE1 in the dual problem. Also, the influence of values of d, the degree of the FEM basis in the element equilibrium problem. Two extreme values are tested, $d = 2$ and 6, with linear weight function associated to node 1 of the TE1 in the QI definition.

The results consistently show good approximation with the exact solution of the values of QI obtained from the average of the error bounds, for all values of D. It must be noticed that the values shown for $D = 2$ means that the basis used in the dual problem is the same used in the primal problem. Even in this case we can obtain meaningful results for the bounds and for the average estimate of the QI. This case corresponds to the most inexpensive option for the analysis, that is, the stiffness matrices for both primal and dual problems are the same and need to be factorized only once.

Figure 5 searches the asymptotic behavior of the bounds with d, where, differently from the other graphs, the equilibrium problem on each element is solved with polynomials of degrees extending up to $d = 9$. It is seen that a basis degree d equal to about 6 is reasonably sufficient to approximate well the equilibrated stress field in each element. Results for $d = 1$ are only illustrative, because the primal problem is solved in this case using GFEM of degree $p = 2$, and the equilibrated resultant forces on each element's boundary represent a loading which is clearly too complex to to allow a FEM solution that approximates well an equilibrated stress field using a FEM basis of degree only $d = 1$. Thus, the bounds are set much more apart than the cases of larger values of d. However, even in this case, the average value is well behaved.

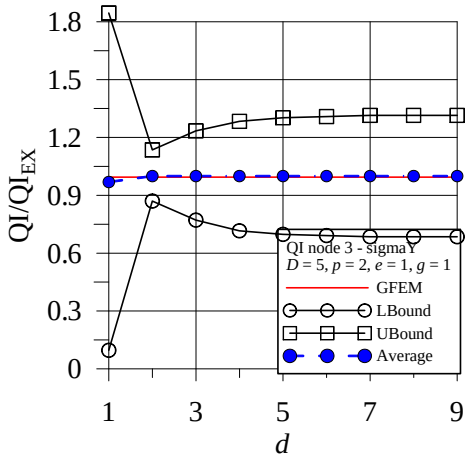

Figure 5: QI defined with node 3 weight function over the target element TE1, for σ_y, versus the degree d of the FEM basis in the equilibrium problem. $D = 5$**,** $p = 2$**.**

The range of the error bounds are equal to the product $E\ \tilde{E}$ between the CRE energy norms of error estimates in the primal and dual problems respectively. Figure 6 show the evolution of both contributions along the basis degree of the equilibrium problem d. For the dual problem, the loading is for a QI associated with a linear weight function for local node 3 of the TE1. On the right vertical axis the errors are normalized by $E^P_{CREAver}$ and $E^D_{CREAver}$ respectively for tha primal and dual problems, which are defined as

$$E^P_{CREAver} = E_{CRE}(\hat{\mathbf{u}}_h, \hat{F}^m_h), \quad \text{and} \quad E^D_{CREAver} = E_{CRE}\left(\widehat{\mathbf{u}}_h, \widehat{\tilde{F}}^m_h\right), \tag{34}$$

where E_{CRE} in both cases are computed according to (7) and $\hat{F}^m_h$ and $\widehat{\tilde{F}}^m_h$ are the averages of equilibrated resultant forces computed according to (14) as

$$\hat{F}^m_h = \frac{1}{2}\left(\widehat{F}_h + \mathbf{C}\boldsymbol{\Sigma}(\widehat{\mathbf{u}}_h)\right), \quad \text{and} \quad \widehat{\tilde{F}}^m_h = \frac{1}{2}\left(\widehat{\tilde{F}}_h + \mathbf{C}\boldsymbol{\Sigma}(\widehat{\tilde{\mathbf{u}}}_h)\right). \tag{35}$$

The errors for primal and dual problems are several orders of magnitude different. For example, for $d = 9$, we obtain the following values for the primal problem: $E^P_{CRE} = 8.11 \cdot 10^{-4}\sqrt{\text{Nmm}}$, $E^P_{CREAver} = 11.9 \cdot 10^{-3}\sqrt{\text{Nmm}}$, such that $E^P_{CRE}/E^P_{CREAver} = 0.0677$, and for the dual problem: $E^D_{CRE} = 63.5\sqrt{\text{Nmm}}$, $E^D_{CREAver} = 78.6\sqrt{\text{Nmm}}$, such that $E^D_{CRE}/E^D_{CREAver} = 0.807$.

Considering the $E_{CREAver}$ as reference, it can be seen that the global GFEM discretization error in the dual problem is one order of magnitude larger than the primal one. The degree used in the GFEM basis in the dual problem is $D = 5$ only over the region Ω_1 around the target element TE1 (the gray region shown in Figure 1). The global error depends on the discretization in the region around the target element where the localized dual loading is applied. In Figure 7 we test the effect of applying especial enrichment over regions of two sizes around the element. The first is on the nodes of the Ω_1 and in the second case, the enrichment of degree D is applied only on the three nodes of the TE1. In the second case, the transition region is composed by all elements around TE1, that is, on this vicinity, the basis functions of approximation are not able to represent completely polynomials of degree D, because the nodes that are not in TE1 are enriched only with the polynomials of the primal problem, in the present case, of degree $p = 2$. First, we observe the higher rate of convergence when the enriched region is larger. Second, the relative errors are smaller, as expected.

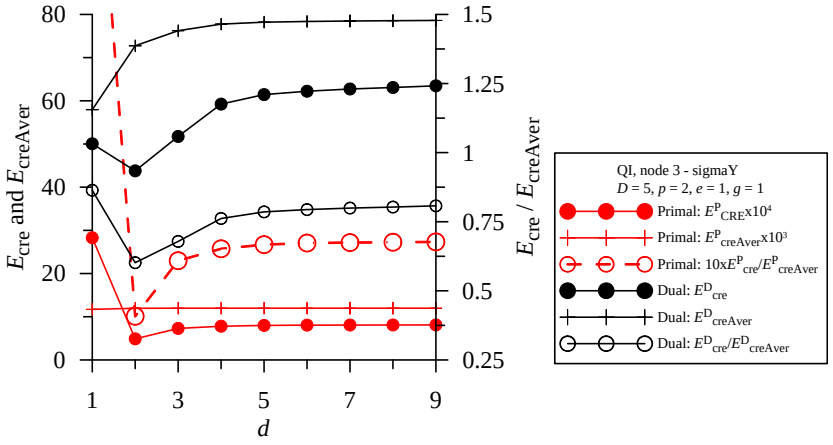

Figure 6: CRE energy norms of error estimates for primal and dual problem associated with node 3 of the target element TE1, for σ_y, versus the degree d of the FEM basis in the equilibrium problem. $D = 5$, $p = 2$.

The lower and upper bounds for the set of QI´s issued from the three linear weighting functions associated with the target element are used to obtain pointwise values of the stress

component used in the definition of the QI, σ_y in the present case, over all points on the target element. We obtain (a) an estimate from the lower bound, that is, $\sigma_y(\mathbf{x}) = U_Q(\mathbf{x}) + \sigma_h(\mathbf{x})$, (b) an estimate from the upper bound, that is, $\sigma_y(\mathbf{x}) = L_Q(\mathbf{x}) + \sigma_h(\mathbf{x})$, (c) and, similarly, an estimate from the average of the bounds: $\sigma_y(\mathbf{x}) = \frac{1}{2}[L_Q(\mathbf{x}) + U_Q(\mathbf{x})] + \sigma_h(\mathbf{x})$, for $\forall \mathbf{x} \in \Omega_{TE1}$.

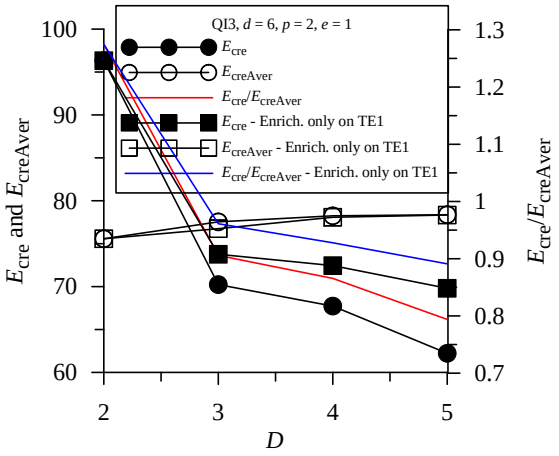

Figure 7: CRE energy norms of error estimates for dual problem with QI associated with node 3 of the target element TE1, for σ_y. $d = 6$, $p = 2$, $g = 1$. Results for enrichment of dual problem over subdomain Ω_1 and over only element TE1.

9 CONCLUSIONS

A formulation for error estimation in quantity of interest was developed for the bending problem of composite laminated plates based on the Mindlin-Reissner kinematic model discritized by the Generalized Finite Element Method (GFEM). The error estimation is based on upper bounds in energy norm obtained by the basic CRE (Constitutive Relation Error) framework, that is, the estimate is obtained from a statically admissible stress field computed at element level in a Neumann problem where the element boundary forces are equilibrated. For a GFEM basis with high order enrichment, an adequate procedures had to be sought, consisting in the use of one single higher order finite element, based on displacement FEM, to obtain an approximation to the equilibrated field. The formulation is implemented for arbitrary degree of the basis, which allows an arbitrarily close approximation to the equilibrium condition.

The strategy used to sharpness the QI's error bounds was to model the dual problem using a GFEM *p*-enrichment with polynomial enrichment functions over a local region around the dual loading in the same mesh used in the primal numerical problem. Therefore the additional cost to solve the dual problem with respect to the primal one is small, because the factorized initial stiffness matrix can be re-used. The GFEM *p*-enrichment is implemented in a simple and straightforward way, as opposed to some other possible forms of enrichment, e.g. local *h*-refinement or a sub-domain approach. Results showed that this strategy can be effective to obtain accurate estimates for the average of the quantity of interest obtained from the estimates of upper and lower bounds. The bounds themselves are not ideally sharp. The size of their range is due to the difficulty of the *p*-enrichment to approximate well concentrated solutions with high gradients.

Besides the evaluation of the p-enrichment in the dual problem, the sensibility of the Constitutive Relation Error (CRE) method for a posteriori error estimate in certain types of Quan-

tities of Interest was also evaluated with respect to the degree d of the Lagrangian triangular function basis used in the FEM for the equilibrium problem in each isolated element problem. Results show that for all degrees above 1, the estimated energy norm of the error is larger than the exact error norm. Small values of d seems to generate, in this specific problem, more accurate approximations, although it is known that only the asymptotic value is a guaranteed upper bound. The asymptotic value for the estimated error norm seems to be reasonable attained with degrees $d = 4$ or 5. It is worth noticing that the use of a single displacement based finite element of high order seems to be very effective to obtain acceptable admissible stresses to be used in the upper bound computation. Although the displacement based FE cannot produce perfectly equilibrated stress fields, a simple hierarchic coding can provide stress fields arbitrarily close to the local equilibrium in the element. This seems to be much simpler option than the hybrid formulations.

All tests were performed with QI's defined as moments of a stress components over a given element. These moments were not only average, as usually it is done, but defined from polynomial weight functions of arbitrary degree g. The tests were done with weights of degrees zero and one.

Acknowledgments

Clovis S. de Barcellos and Paulo T. R. Mendonça gratefully acknowledge the financial support provided by the Brazilian government agency CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico) for this research, under research grants 304.702/2013-7 and 304.698/2013-0, respectively.

REFERENCES

Barros, F. B., de Barcellos, C. S., Duarte, C. A. & Torres, D. A., 2013. Subdomain-based error techniques for generalized finite element approximations of problems with singular stress fields. *Computational Mechanics*, 52(6), pp. 1395-1415.

Benoit, C., Coorevits, P. & Pelle, J. P., 1999. Error estimation for plate structures: application using the DKT element. *Engineering Computations*, 16(5), pp.584-600.

Carstensen, C. & Schöberl, J., 2006. Residual-based a posteriori error estimate for a mixed Reissner-Mindlin plate finite element method. *Numerische Mathematik*, 103(2), pp.225-250.

Da Veiga, L. B., Niiranen, J. & Stenberg, R., 2013. A posteriori error analysis for the postprocessed MITC plate elements. *SIAM Journal on Numerical Analysis*, 51(1), pp.1-23.

Díez, P., Jos Rdenas, J. & Zienkiewicz, O. C., 2007. Equilibrated patch recovery error estimates: simple and accurate upper bounds of the error. *International Journal for Numerical Methods in Engineering*, 69(10), pp.2075-2098.

Dobyns, A. L., 1981. Analysis of simply-supported orthotropic plates subject to static and dynamic loads. *AiAA Journal*, 19(5), pp.642-650.

Ladevèze, P., 1975. Comparison of models of continuum media. *These de doctorat d'etat*, Université Pierre et Marie Curie, Paris.

Ladevèze, P. & Chamoin, L., 2010. Calculation of strict error bounds for finite element approximations of non-linear pointwise quantities of interest, *International Journal for Numerical Methods in Engineering*, 84, 1638-1664.

Ladevèze, P., & Pelle, J. P., 2005. *Mastering calculations in linear and nonlinear mechanics*. F. F. Ling, E. F. Gloyna, & W. H. Hart (Eds.). New York, NY: Springer. Berlin.

Ladevèze, P. & Rougeot, P., 1997. New advances on a posteriori error on constitutive relation in fe analysis. *Computer Methods in Applied Mechanics and Engineering*, 150(1), pp.239-249.

Mendonça, P. T. R., de Barcellos, C. S., & Torres, D. A. F., 2011. Analysis of anisotropic Mindlin plate model by continuous and non-continuous GFEM. *Finite Elements in Analysis and Design*, v. 47, n. 7, p. 698-717.

Mendonça, P. T. R., de Barcellos, C. S., & Torres, D. A. F., 2013. Robust Ck/C0 generalized FEM approximations for higher-order conformity requirements: Application to Reddy's HSDT model for anisotropic laminated plates. *Composite Structures*, v. 96, p. 332-345.

Oden, J. T. & Prudhomme, S., 2002. Estimation of modeling error in computational mechanics, *Journal of Computational Physics*, 182, 496-515.

Parés, N., Díez, P., & Huerta, A., 2006. Subdomain-based flux-free a posteriori error estimators. *Computer Methods in Applied Mechanics and Engineering*, 195(4), pp.297-323.

Pelle, J.P., 1995. Sur la maîtrise des calculs Eléments Finis: état actuel et persperctives. *Paris Actes du deuxième colloque national en calcul des structures*, Giens, França, 43-54, 1995.

Prager, W. & Synge, J.L., 1947. Approximations in elasticity based on the concept of function space. *Quarterly of Applied Mathematics*, 5(3), pp.241-269.

Zienkiewicz, O. C. and Zhu, J. Z., 1992. The superconvergent patch recovery and *a-posteriori* error estimates. Part 1: The recovery technique. *International Journal for Numerical Methods in Engineering*, 33(7), pp.1331-1364.