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ISOGEOMETRIC ANALYSIS APPLIED TO FREE VIBRATION ANALYSIS OF PLANE STRESS AND PLANE STRAIN STRUCTURES

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Abstract. *Actually Isogeometric Analysis (IGA) have proven high accuracy and efficacy in dynamical problems, openning possibilities to improve the traditional FEM models. The aim of this paper is to test the response of IGA for free vibration problems of plane stress and structures. Based on numerical applications, IGA models have their convergence and accuracy checked and compared with those developed in FEM and GFEM. The results shows high accuracy for IGA models, and reinforce its way as a promising tool.*

Keywords: *Isometric Analysis, Free Vibration, Plane Stress*

1 INTRODUCTION

Since the introduction to Isogeometric Analysis (IGA) by Hughes et al. (2005) a large amount of applications and improvements were done in the context of numerical methods, where researches were motivated by a couple of advantages which the method promised. In dynamical analysis scenario, IGA presented high accuracy over Finite Element Method (FEM) (Cottrell et al. 2006) in the whole free vibration frequency sample.

At the same time in the extended versions of the classical FEM, mostly those based in Partition of Unity Method (PUM) (Melenk & Babuska 1996), a couple of steps forward were done in dynamical analysis. Concerning this work scope, highlight the works of Arndt et al. (2010) and Torii & Machado (2012) whose applied Generalized Finite Element Method (GFEM) for the free vibration problem of bars and trusses. Naturally other kind of advances in the called "Enriched Methods" based in PUM were performed, as the Stable GFEM (Babuška & Banerjee 2012) and the recent Orthonormalized GFEM (Sillem et al. 2014).

Due to the relative success of IGA from the results of the frequency error spectra for the free vibration problem of straight bars and beams (Cottrell et al. 2009, Rauen 2014), this work aims to extend the vibration tests to plain stress structures. The results of IGA, presented in form of free vibration frequencies and error spectra, are compared with those developed by Torii & Machado (2012) and Torii (2012) for GFEM and classical FEM.

2 ISOGEOMETRIC ANALYSIS

Isogeometric Analysis is a FEM-like numerical method which reformulated the treatment of object geometry and mesh questions. Aiming to solve FEM mesh bottlenecks, which demands high computational costs, IGA works by means of NURBS (*Non Uniform Rational B-Splines*), which allow to connect CAD environment with FEA, since those functions are the same.

IGA follows the opposite way of Isoparametric Concept. Since FEM turns to find a set of functions to describe the mathematical problem, IGA aims to find a set of NURBS capable to describe object geometry perfectly (Cottrell et al. 2009).

2.1 NURBS Functions

NURBS is a family of B-Splines functions. It follows the recursive scheme of construction of Cox and de-Boor (De-Boor 1972, Cox 1972). This formulation constructs a base of n B-Splines with order p, where its behaviour depends on the called knot vector Ξ. The knot vector consists in a set of non decreasing coordinates, called knots. Given a polynomial degree p, a number of n shape functions and a knot vector $\Xi = \{\xi_1, \xi_2, \ldots, \xi_{n+p+1}\}$, B-Splines basis functions are defined by:

$$N_{i,0}(\xi) = \begin{cases} 1 & \text{if } \xi_i \leq \xi < \xi_{i+1}, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

for $p = 0$ and

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi), \tag{2}$$

for $p \geq 1$.

For IGA, a basic set of NURBS shape functions is defined by repeating the edge knots $p+1$ times. Some relevant NURBS properties are described in Hughes et al. (2005) and Piegl & Tiller (1997) which extensively contribute to the performance and optimization of IGA implementations. Figure 1 shows an example of NURBS shape functions with parameters $p = 2$, $n = 6$ and $\xi = \{0, 0, 0, 0.25, 0.5, 0.75, 1, 1, 1\}$.

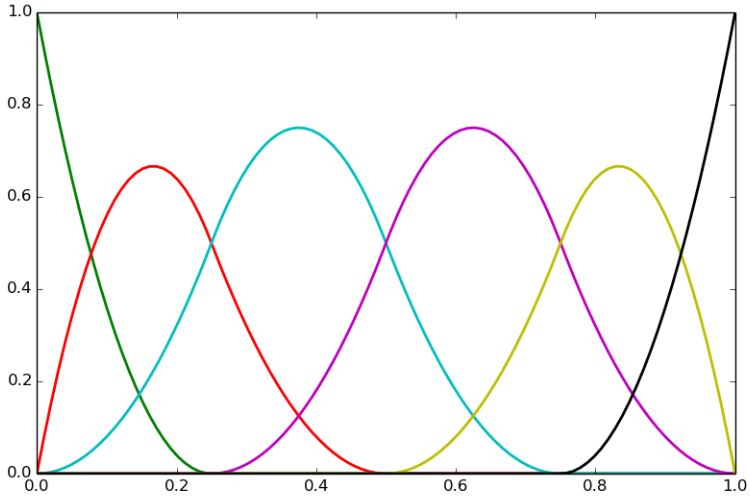

Figure 1: NURBS Shape Functions

2.2 NURBS Surfaces

NURBS surfaces are defined by functions product. Given $N_{i,p}(\xi)$ a set of NURBS with order p, n functions and knot vector Ξ, and $M_{j,q}(\eta)$ a set of NURBS defined by order q, m shape functions and knot vector H, the surface equation is defined by:

$$\tilde{N}_{i,j;p,q}(\xi, \eta) = N_{i,p}(\xi) M_{j,q}(\eta). \tag{3}$$

Figure 2 shows an example of a set of NURBS surfaces with the parameters $p = q = 2$, $n = m = 4$ and $\Xi = H = \{0, 0, 0, 0.5, 1, 1, 1\}$.

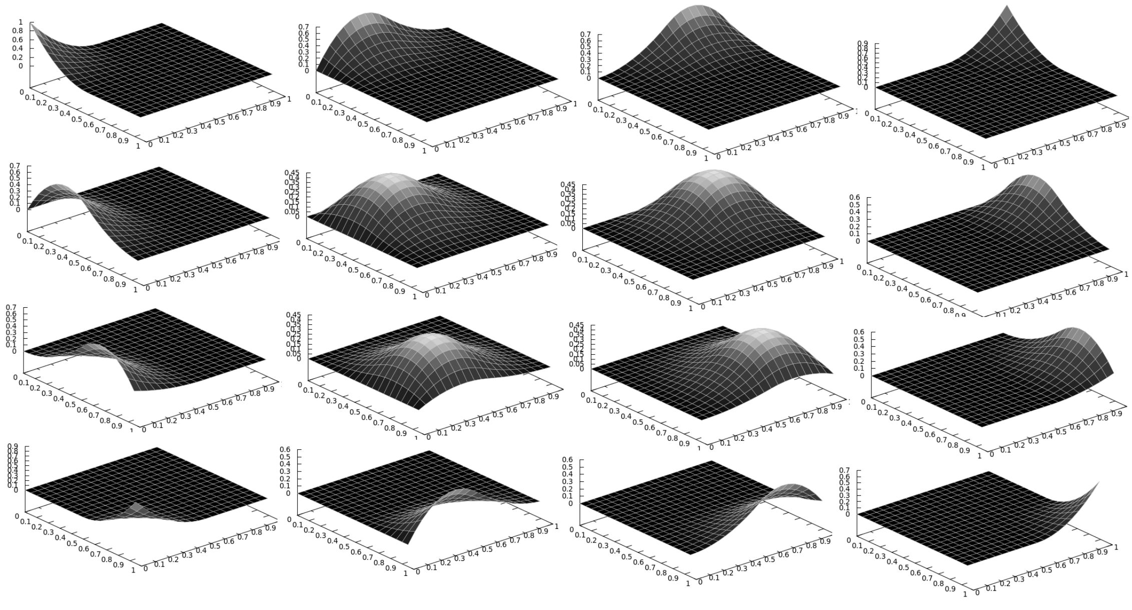

Figure 2: NURBS Surfaces

2.3 IGA Refinements

In the viewpoint of shape functions, IGA refinement could be seen as a set of modification in the functions parameters. Basically the innput parameters Ξ, n and p are modified and a new set of shape functions is created. Different kind of modifications describes the different kinds of refinements.

Isogeometric *h* refinement consists to change n and Ξ parameters only. With the increasing in the number of shape functions n there's a need also to add knot in Ξ. This results in a increasing in the number of shape functions with the same order p. Considering frequency error spectra for the free vibration of rods and beams (Cottrell et al. 2006), is proven that *h* refinements does not change the behaviour of normalized spectrum curves.

Cottrell et al. (2007) define the Isogeometric p refinement as the order increasing with continuity maintained. The number of shape functions n is also increased, but an important fact is related with knot vector: the multiplicity of the whole set of knots is also increased. Details of p refinement implementation are given by Cottrell et al. (2007) and Cottrell et al. (2009). Some comparisons isogeometric p refinement and other refinements developed by Rauen et al. (2013).

NURBS shape functions allows to control their continuities with parameters p and the multiplicity of knots. The concept of the k refinement is to increase polynomial degree without increase interior knots multiplicity. This gives a high continuity in element domain (Cottrell et al. 2007). Convergence rates in k refinement were proven higher than p refinement (Rauen et al. 2013, Rauen 2014), due to inscrease continuity and shape functions smoothness with a lower number of shape functions.

3 ENRICHED METHODS

Every method which uses additional functions to increase the shape functions space of the classical FEM could be called enriched method (Arndt et al. 2011, Arndt 2009).

The enriched methods presents the approximate solution given by:

$$u_h^e = u_{FEM}^e + u_{ENRICHED}^e \tag{4}$$

In matrix form, Eq. (4) defined by:

$$u_h^e = N^T q + \phi^T \bar{q} \tag{5}$$

where u_{FEM}^e is the FEM displacement field based on nodal degrees of freedom, $u_{ENRICHED}^e$ is the enriched displacement field based on field degrees of freedom, q is the FEM degrees of freedom vector and $\bar{q}$ is the field degrees of freedom vector. The vectors N and ϕ contain the classical FEM shape functions and the enriched shape functions, respectively. The vectors ϕ and $\bar{q}$ are defined by:

$$\phi^T(\xi) = [\ F_1 \quad F_2 \quad \ldots \quad F_r \quad \ldots \quad F_n\ ] \tag{6}$$

$$\bar{q} = [\ c_1 \quad c_2 \quad \ldots \quad c_r \quad \ldots \quad c_n\ ] \tag{7}$$

where F_r are the enrichment functions, c_r are the field degrees of freedom. Each enriched method is defined by a different set of shape functions.

The Composite Element Method (CEM) (Zeng 1998a) uses the general analytical solution of vibration problems as enrichment functions, trigonometric functions appears to enrich the basis functions space. GFEM uses the PUM-based functions, leading to a set of methods more flexible to describe the basis functions space of a general phenomenon.

4 Plane Stress Variational Formulation

The free vibration of the plane stress phenomena is given by a set of differential equations, described as:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = \rho \frac{\partial^2 \bar{u}}{\partial t^2} \tag{8}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = \rho \frac{\partial^2 \bar{v}}{\partial t^2} \tag{9}$$

where ρ is the specific mass, $\sigma_x, \sigma_y, \tau_{xy}$ are the stresses and $\bar{u}, \bar{v}$ are the horizontal and vertical displacements.

The eigenvalue formulation is obtained by the description of eqs.(9) in terms of strain and then to apply variational techniques. Formulation details of the free vibration of plane stress element are developed by Reddy (1993). Those applications, followed by substitution of the basic solution, leads to the expression:

$$h \iint_\Omega \left[\frac{\partial w_1}{\partial x} \left(c_{11} \frac{\partial u}{\partial x} + c_{12} \frac{\partial v}{\partial y} \right) + \frac{\partial w_1}{\partial y} c_{33} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] dx\, dy - \rho \lambda \iint_\Omega w_1 u \, dx\, dy = 0, \tag{10}$$

$$h \iint_\Omega \left[\frac{\partial w_2}{\partial x} c_{33} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + \frac{\partial w_1}{\partial y} \left(c_{12} \frac{\partial u}{\partial x} + c_{22} \frac{\partial v}{\partial y} \right) \right] dx\, dy - \rho \lambda \iint_\Omega w_2 v \, dx\, dy = 0. \tag{11}$$

where h is the element thickness, ω_1, ω_2 are weighting functions and $c_{11}, c_{12}, c_{22}, c_{33}$ are the components c_{ij} of the constitutive matrix, which is given by:

$$\mathbf{C} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}. \tag{12}$$

The application of Galerkin's Method turns the system an aproximate generalized eigenvalue problem, given by:

$$\mathbf{K}\Delta^h = \lambda^h \mathbf{M}\Delta^h \tag{13}$$

where

$$\Delta^h = \begin{Bmatrix} u^h \\ v^h \end{Bmatrix} \tag{14}$$

and u^h, v^h are the horizontal and vertical plane stress displacements.

The matrixes $\mathbf{K}$ and $\mathbf{M}$ are given by the expressions:

$$\mathbf{K} = h\int_\Omega \mathbf{B^T C B}\, dx\, dy \tag{15}$$

$$\mathbf{M} = \rho h\int_\Omega \mathbf{H^T H}\, dx\, dy \tag{16}$$

and the matrixes $\mathbf{H}$ and $\mathbf{B}$ being defined as:

$$\mathbf{H}^T = \begin{bmatrix} \tilde{N}_{1,1:p} & 0 & \tilde{N}_{1,2:p} & 0 & \dots & \tilde{N}_{N,M:p} & 0 \\ 0 & \tilde{N}_{1,1:p} & 0 & \tilde{N}_{1,2:p} & \dots & 0 & \tilde{N}_{N,M:p} \end{bmatrix}, \tag{17}$$

and

$$\mathbf{B} = \mathbf{DH} \tag{18}$$

where $\mathbf{D}$ is an operator given by:

$$\mathbf{D} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \tag{19}$$

5 NUMERICAL EXAMPLES

5.1 Unitary Square Steel Plate

The example developed above and ilustrated by fig. 3 consists in a stell plate with $L_x = L_y = 1m$ and $h = 0.1m$. Material properties are: $\rho = 8000kg/m^3$, $E = 210GPa$ and $\nu = 0.3$.

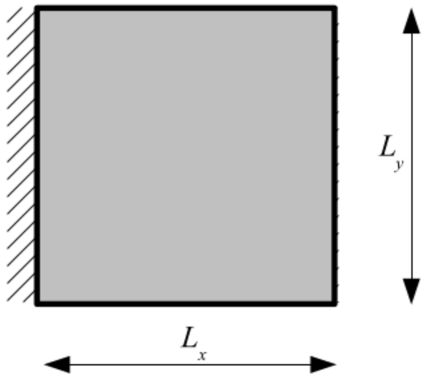

Figure 3: Square Steel Plate (Torii 2012)

Natural Vibration Frequencies and Errors

Table 3 shows the percentual errors for models developed in Hierarchical FEM (HFEM) and GFEM by Torii (2012) and IGA with polynomial degrees $p = 3$ and $p = 5$. The results used to determine the errors is a refined model developed in HFEM with degree $p = 9$ also, by Torii (2012).

Table 1: Square Steel Plate Vibration Frequencies

i	ω_i (rad/s)	HFEM	GFEM π	IGA $p = 3$	IGA $p = 5$
1	$3372,13$	$0,057055$	$0,063529$	$0,035073$	$0,010609$
2	$8092,72$	$0,018804$	$0,020749$	$0,011647$	$0,003539$
3	$9079,09$	$0,010432$	$0,013191$	$0,005292$	$0,001138$
4	$14427,23$	$0,002301$	$0,002959$	$0,001192$	$0,000260$
5	$15558,23$	$0,030022$	$0,034655$	$0,017811$	$0,005016$
6	$16511,96$	$0,000493$	$0,000658$	$0,000291$	$0,000067$
7	$20812,77$	$0,015192$	$0,017391$	$0,006618$	$0,000988$
8	$21911,61$	$0,036135$	$0,042266$	$0,020378$	$0,005238$
9	$24194,74$	$0,012133$	$0,013001$	$0,005003$	$0,000499$

Frequency Spectrum

Figure 4 shows graphically the same results shown in Table 3.

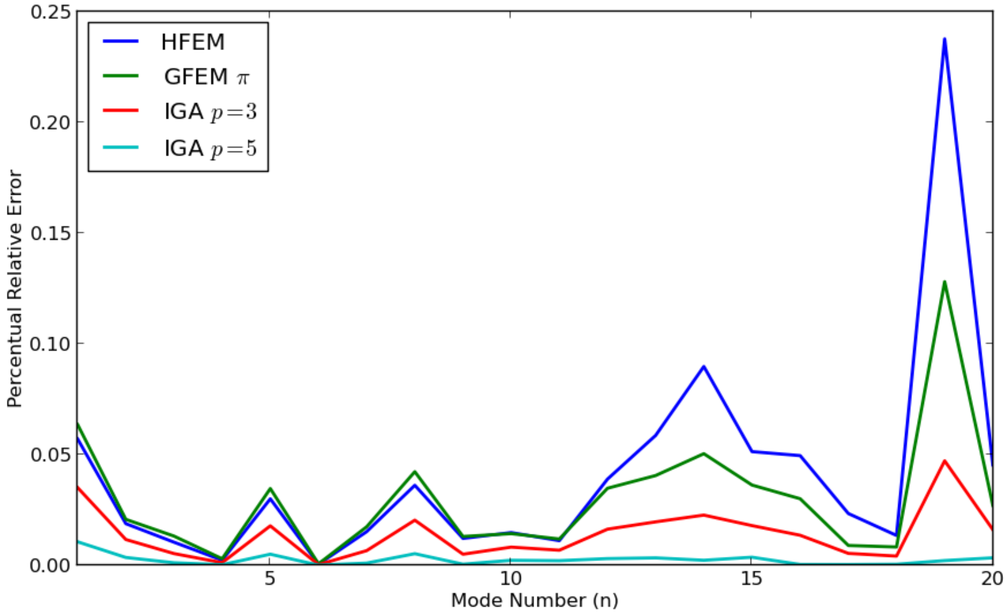

Figure 4: Steel Square Plate Frequency Error Spectra

The plane state frequency spectrum shows a set of more accurate curves for IGA models followed, in accuracy, by GFEM and HFEM.

6 CONCLUSIONS

This work aimed to test the efficiency of Isogeometric Analysis as approach to the free vibration problem of plane stress. The results show accurate behaviour of IGA shape functions if compared with FEM and GFEM. Concerning bidimensional problems like plane state, IGA presented some advantages mostly by the fact of global directly defined NURBS functions in physical space. Results shows highlights in accuracy, giving feasibility in future IGA modelling.

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