CONVERGENCE PROOF OF A TEMPERATURE-BASED FINITE ELEMENT FORMULATION FOR TRANSIENT HEAT TRANSFER

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Abstract. *A finite element is developed to discretize spatially one-dimensional transient heat conduction problems in both space and time. Stability of the recursive discretized equation is* *proved using the eigenvalues of the amplification matrix. Convergence, in this case, stems naturally from the formulation consistency of the developed finite element. Numerical experiments demonstrate that refined models are in close agreement with the exact solution.*

Keywords: *Convergence, Eigenvalue bound, Finite element, Heat conduction*

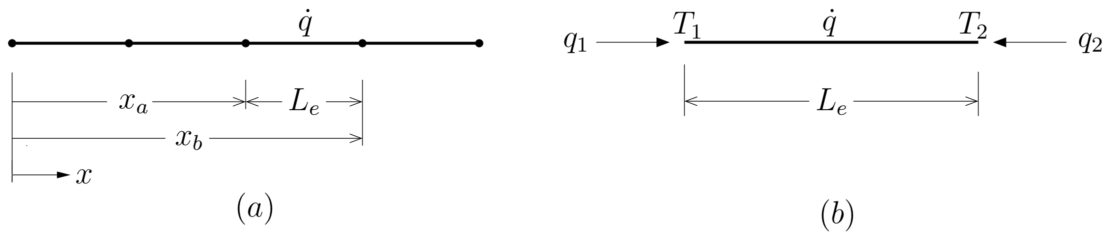

Figure 1: (a) A sketch of the finite element mesh in the x direction (t direction omitted); (b) a typical two-node element

1 INTRODUCTION

The diffusion equation

$$\rho c \frac{\partial T}{\partial t} - \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) - \dot{q} = 0 \qquad 0 < x < L \qquad t > 0, \tag{1}$$

subjected to boundary conditions at $x = 0$ and $x = L$ and initial condition at $t = 0$, can be used to describe a variety of physical phenomena, such as heat conduction (Bergman et al., 2011), soil consolidation (Lambe & Whitman, 1969), drug dispersal in human tissues (Barry, 2002), and so on. The application of the equation to these different topics should of course be preceded by the attribution of different meaning to the quantities present therein. In the particular case of spatially one-dimensional transient heat conduction, $T(x, t)$ is the temperature at time t and position x, and $\dot{q}$ is the rate of energy generation per unit volume. The material properties are provided by the thermal conductivity k, the mass density ρ and the specific heat c.

Temperature-based finite element models, related to the solution of (1), are traditionally the most currently used in heat conduction analysis because of their simplicity and general effectiveness. A simple finite element of this type, with linear interpolation of the temperature in both space and time, is developed in this paper. Thus, no additional technique such as finite difference would be required for time-domain integration. The discrete problem has its convergence proved analytically by invoking a theorem due to Irons & Treharne (1971) and another due to Lax & Richtmyer (1956). A numerical example illustrates this convergent behavior.

2 FINITE ELEMENT FORMULATION

Figure 1 shows the problem space domain divided into a number of finite elements, and a typical element with endpoints at $x = x_a$ and $x = x_b$ isolated from the mesh. The element length in the x direction is given by $L_e = x_b - x_a$. In order to develop a finite element discretization in space, equation (1) is enforced to be satisfied in $x_a \le x \le x_b$ by means of

$$\int_{x_a}^{x_b} \left[\rho c \frac{\partial T}{\partial t} - \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) - \dot{q}\right] \delta T \, dx = 0 \tag{2}$$

where δT is the first variation of the temperature. In view of the integration by parts

$$\int_{x_a}^{x_b} \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) \delta T \, dx = k \frac{\partial T}{\partial x} \delta T \Big|_{x_a}^{x_b} - \int_{x_a}^{x_b} k \frac{\partial T}{\partial x} \frac{\partial \, \delta T}{\partial x} dx, \tag{3}$$

equation (2) reduces to

$$\int_{x_a}^{x_b} \left[\rho c \frac{\partial T}{\partial t} \delta T + k \frac{\partial T}{\partial x} \frac{\partial \, \delta T}{\partial x} - \dot{q} \, \delta T \right] dx - q_1 \, \delta T(x_a, t) - q_2 \, \delta T(x_b, t) = 0 \tag{4}$$

with the inclusion of the heat flow $q_1 = -k \, \partial T(x_a, t)/\partial x$ and $q_2 = k \, \partial T(x_b, t)/\partial x$ into the element at the nodes. Equation (4) is preferable to equation (2) for finite element discretization in space because has lower order derivative with respect to x (first-order instead of second-order derivative), so requiring simpler interpolations. Another important benefit is that equation (4) also enforces eventual boundary conditions in heat flow to be satisfied, so that we need not worry about constructing interpolations to satisfy those conditions.

After identifying equation (4) as our starting point for the finite element discretization in space, we proceed assuming that the temperature distribution over $x_a \leq x \leq x_b$ varies, at any instant of time, linearly between the nodal values $T_1(t)$ at $x = x_a$ and $T_2(t)$ at $x = x_b$:

$$T(x,t) = \frac{x_b - x}{L_e} T_1(t) + \frac{x - x_a}{L_e} T_2(t). \tag{5}$$

This linear interpolation is the simplest choice for which any value of constant temperature or its derivative with respect to x can be represented within the element, as well as the continuity of the temperature can be represented between elements (Fish & Belytschko, 2007). Substitution of (5) into (4) yields

$$\begin{aligned} &\left[\frac{\rho c L_e}{6} \left(2\dot{T}_1 + \dot{T}_2 \right) + \frac{k}{L_e} \left(T_1 - T_2 \right) - \dot{q}_1 - q_1 \right] \delta T_1 \\ + &\left[\frac{\rho c L_e}{6} \left(\dot{T}_1 + 2\dot{T}_2 \right) + \frac{k}{L_e} \left(T_2 - T_1 \right) - \dot{q}_2 - q_2 \right] \delta T_2 = 0 \end{aligned} \tag{6}$$

where $\dot{T}_1$ and $\dot{T}_2$ refers to dT_1/dt and dT_2/dt, respectively, and

$$\dot{q}_1 = \int_{x_a}^{x_b} \frac{x_b - x}{L_e} \dot{q} \, dx \qquad \dot{q}_2 = \int_{x_a}^{x_b} \frac{x - x_a}{L_e} \dot{q} \, dx \tag{7}$$

are the equivalent nodal values to the distributed rate of energy generation $\dot{q}$. If we admit that δT_1 and δT_2 are arbitrary and linearly independent, we can write

$$\begin{aligned} &\frac{\rho c L_e}{6} \left(2\dot{T}_1 + \dot{T}_2 \right) + \frac{k}{L_e} \left(T_1 - T_2 \right) - \dot{q}_1 - q_1 = 0 \\ &\frac{\rho c L_e}{6} \left(\dot{T}_1 + 2\dot{T}_2 \right) + \frac{k}{L_e} \left(T_2 - T_1 \right) - \dot{q}_2 - q_2 = 0 \end{aligned} \tag{8}$$

or, in matrix form,

$$\mathbf{A} \, \dot{\mathbf{T}} + \mathbf{B} \, \mathbf{T} = \mathbf{q} \tag{9}$$

with

$$\mathbf{T} = \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} \; \mathbf{A} = \frac{\rho c L_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \; \mathbf{B} = \frac{k}{L_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \; \mathbf{q} = \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix} + \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}. \tag{10}$$

The set of ordinary differential equations (9) represents the space discretization of the partial differential equation (1) over $x_a \leq x \leq x_b$ of an element domain, which also includes the heat flow into the element at the nodes, using the finite element method.

Typically, the set of equations (9) is further approximated in time using a finite difference scheme for the time derivative. The most commonly used scheme is the α-family of approximation (Reddy, 2006), in which a weighted average of the time derivatives $\dot{\mathbf{T}}$ at two consecutive time steps $t = t_n$ and $t = t_{n+1}$ is approximated by linear interpolation of the values of $\mathbf{T}$ at the two steps:

$$\dot{\mathbf{T}} \approx (1-\alpha)\dot{\mathbf{T}}(t_n) + \alpha\dot{\mathbf{T}}(t_{n+1}) \approx \frac{\mathbf{T}(t_{n+1}) - \mathbf{T}(t_n)}{t_{n+1} - t_n} \qquad 0 \leq \alpha \leq 1. \tag{11}$$

This equation can be expressed alternatively as

$$\mathbf{T}^{\Delta t} \approx \mathbf{T}^0 + \Delta t \left[(1-\alpha)\dot{\mathbf{T}}^0 + \alpha\dot{\mathbf{T}}^{\Delta t}\right] \qquad 0 \leq \alpha \leq 1 \tag{12}$$

where $\Delta t = t_{n+1} - t_n$, and $\mathbf{T}^0$ and $\mathbf{T}^{\Delta t}$ are the nodal values at times t_n and t_{n+1}, respectively.

Equation (9) at time steps $t = t_n$ and $t = t_{n+1}$ reads

$$\begin{aligned} \mathbf{A}\dot{\mathbf{T}}^0 + \mathbf{B}\mathbf{T}^0 &= \mathbf{q}^0 \\ \mathbf{A}\dot{\mathbf{T}}^{\Delta t} + \mathbf{B}\mathbf{T}^{\Delta t} &= \mathbf{q}^{\Delta t}, \end{aligned} \tag{13}$$

where $\mathbf{A}$ and $\mathbf{B}$ are supposed to be time independent. Elimination of $\dot{\mathbf{T}}^0$ and $\dot{\mathbf{T}}^{\Delta t}$ from (12) by means of (13) yields

$$(\mathbf{A} + \alpha\,\Delta t\,\mathbf{B})\,\mathbf{T}^{\Delta t} = [\mathbf{A} - (1-\alpha)\,\Delta t\,\mathbf{B}]\,\mathbf{T}^0 + \Delta t\left[\alpha\mathbf{q}^{\Delta t} + (1-\alpha)\mathbf{q}^0\right], \tag{14}$$

which represents the time discretization of (9) using the α-family of approximation.

In order to develop an alternative discretization in time using the finite element method, equation (9) is first enforced to be satisfied in $t_n \leq t \leq t_{n+1}$ by means of

$$\int_{t_n}^{t_{n+1}} \delta\mathbf{T}^T \left(\mathbf{A}\dot{\mathbf{T}} + \mathbf{B}\mathbf{T} - \mathbf{q}\right) dt = 0. \tag{15}$$

We will assume that the temperature distribution over $t_n \leq t \leq t_{n+1}$ varies linearly between the nodal values $\mathbf{T}^0$ and $\mathbf{T}^{\Delta t}$:

$$\mathbf{T}(t) = \frac{t_{n+1} - t}{\Delta t}\mathbf{T}^0 + \frac{t - t_n}{\Delta t}\mathbf{T}^{\Delta t}. \tag{16}$$

As for the space interpolation (5), the above time interpolation is the simplest one to be used. For a step-by-step approach of the solution in the t direction, the temperature is supposed to be known at time t_n ($\delta\mathbf{T}^0 = \mathbf{0}$) and to be determined at time t_{n+1} ($\delta\mathbf{T}^{\Delta t}$ is arbitrary). Substitution of (16) into (15) yields

$$\left(\mathbf{A} + \frac{2\,\Delta t}{3}\mathbf{B}\right)\mathbf{T}^{\Delta t} = \left(\mathbf{A} - \frac{\Delta t}{3}\mathbf{B}\right)\mathbf{T}^0 + 2\int_{t_n}^{t_{n+1}} \frac{t - t_n}{\Delta t}\mathbf{q}\,dt. \tag{17}$$

If $\mathbf{q}$ is approximately represented in terms of its values $\mathbf{q}^0$ at t_n and $\mathbf{q}^{\Delta t}$ at t_{n+1} using the linear interpolation

$$\mathbf{q}(t) = \frac{t_{n+1} - t}{\Delta t}\mathbf{q}^0 + \frac{t - t_n}{\Delta t}\mathbf{q}^{\Delta t}, \tag{18}$$

expression (17) reduces to

$$\left(\mathbf{A} + \frac{2\,\Delta t}{3}\mathbf{B}\right)\mathbf{T}^{\Delta t} = \left(\mathbf{A} - \frac{\Delta t}{3}\mathbf{B}\right)\mathbf{T}^{0} + \frac{\Delta t}{3}\left(2\mathbf{q}^{\Delta t} + \mathbf{q}^{0}\right). \tag{19}$$

The set of algebraic equations (19) now represents the discretization by the finite element method of the partial differential equation (1) in both space and time over an element domain $x_a \le x \le x_b$ and $t_n \le t \le t_{n+1}$, which also includes the heat flow into the element at the nodes. The element has lengths $L_e = x_b - x_a$ and $\Delta t = t_{n+1} - t_n$ in the x and t directions, respectively. Comparing (19) with (14), we find that the adopted finite element with linear interpolation (16) and approximation (18) is a special case of the α-family of approximation for $\alpha = 2/3$.

3 AMPLIFICATION MATRIX

We write the discretized equation for the domain $0 \le x \le L$ and $t_n \le t \le t_{n+1}$ in the form

$$\mathbf{R}\mathbf{T}_{n+1} = \mathbf{S}\mathbf{T}_n + \mathbf{Q}, \tag{20}$$

where $\mathbf{T}_n$ and $\mathbf{T}_{n+1}$ are the nodal temperature at times t_n and t_{n+1}, respectively, and matrices $\mathbf{R}$, $\mathbf{S}$ and $\mathbf{Q}$ stem from each element contribution (19). Somewhat because the system (20) is "open ended" in the t direction, its solution can be sensitive to small errors introduced, for instance, by the use of finite word length machine. In order to analyze the error propagation, a small error $\boldsymbol{\varepsilon}_0$ is introduced at time $t = 0$ so that equation (20) takes the form

$$\mathbf{R}(\mathbf{T}_{n+1} + \boldsymbol{\varepsilon}_{n+1}) = \mathbf{S}(\mathbf{T}_n + \boldsymbol{\varepsilon}_n) + \mathbf{Q} \tag{21}$$

and $\boldsymbol{\varepsilon}_n$ and $\boldsymbol{\varepsilon}_{n+1}$ stand for the propagated error at times $t = t_n$ and $t = t_{n+1}$, respectively. Subtraction of (20) from (21) yields

$$\mathbf{R}\boldsymbol{\varepsilon}_{n+1} = \mathbf{S}\boldsymbol{\varepsilon}_n \tag{22}$$

or

$$\boldsymbol{\varepsilon}_{n+1} = \mathbf{R}^{-1}\mathbf{S}\boldsymbol{\varepsilon}_n, \tag{23}$$

in which $\mathbf{R}$ is supposed to be nonsingular and $\mathbf{R}^{-1}\mathbf{S}$ is known as the amplification matrix.

Assuming that the amplification matrix is symmetric of order N, its eigenvectors $\mathbf{r}_i$ are all orthogonal and can be used to express the initial error as the linear combination

$$\boldsymbol{\varepsilon}_0 = c_1\mathbf{r}_1 + \cdots + c_N\mathbf{r}_N \tag{24}$$

where c_i are real numbers. Let λ_i be the eigenvalue associated to the eigenvector $\mathbf{r}_i$ so that

$$\mathbf{R}^{-1}\mathbf{S}\mathbf{r}_i = \lambda_i\mathbf{r}_i. \tag{25}$$

From (23) we get

$$\begin{aligned}\boldsymbol{\varepsilon}_1 &= \mathbf{R}^{-1}\mathbf{S}\varepsilon_0 = \mathbf{R}^{-1}\mathbf{S}\left(c_1\mathbf{r}_1 + \cdots + c_N\mathbf{r}_N\right) = c_1\lambda_1\mathbf{r}_1 + \cdots + c_N\lambda_N\mathbf{r}_N \\ \boldsymbol{\varepsilon}_2 &= \mathbf{R}^{-1}\mathbf{S}\varepsilon_1 = \mathbf{R}^{-1}\mathbf{S}\left(c_1\lambda_1\mathbf{r}_1 + \cdots + c_N\lambda_N\mathbf{r}_N\right) = c_1\lambda_1^2\mathbf{r}_1 + \cdots + c_N\lambda_N^2\mathbf{r}_N\end{aligned} \tag{26}$$

or, more generally,

$$\begin{aligned} \boldsymbol{\varepsilon}_n &= c_1\lambda_1^n\mathbf{r}_1 + \cdots + c_N\lambda_N^n\mathbf{r}_N \\ \boldsymbol{\varepsilon}_{n+1} &= c_1\lambda_1^{n+1}\mathbf{r}_1 + \cdots + c_N\lambda_N^{n+1}\mathbf{r}_N. \end{aligned} \tag{27}$$

If $\mathbf{r}_i$ are normalized to unit length, we write

$$\begin{aligned} \boldsymbol{\varepsilon}_n^T\boldsymbol{\varepsilon}_n &= (c_1\lambda_1^n)^2 + (c_2\lambda_2^n)^2 + \cdots + (c_N\lambda_N^n)^2 \\ \boldsymbol{\varepsilon}_{n+1}^T\boldsymbol{\varepsilon}_{n+1} &= \left(c_1\lambda_1^{n+1}\right)^2 + \left(c_2\lambda_2^{n+1}\right)^2 + \cdots + \left(c_N\lambda_N^{n+1}\right)^2 \end{aligned} \tag{28}$$

to obtain a measure of the error propagation from $t = t_n$ to $t = t_{n+1}$ as being

$$\boldsymbol{\varepsilon}_{n+1}^T\boldsymbol{\varepsilon}_{n+1} - \boldsymbol{\varepsilon}_n^T\boldsymbol{\varepsilon}_n = (c_1\lambda_1^n)^2\left(\lambda_1^2 - 1\right) + (c_2\lambda_2^n)^2\left(\lambda_2^2 - 1\right) + \cdots + (c_N\lambda_N^n)^2\left(\lambda_N^2 - 1\right). \tag{29}$$

The recurrence relation (20) is said to be absolutely stable if

$$|\boldsymbol{\varepsilon}_{n+1}| < |\boldsymbol{\varepsilon}_n|, \tag{30}$$

which implies that

$$\boldsymbol{\varepsilon}_{n+1}^T\boldsymbol{\varepsilon}_{n+1} - \boldsymbol{\varepsilon}_n^T\boldsymbol{\varepsilon}_n < 0. \tag{31}$$

This condition is clearly fulfilled by

$$\lambda_i^2 \le 1 \quad \Rightarrow \quad |\lambda_i| \le 1 \qquad i = 1, 2, \ldots, N. \tag{32}$$

For a large system of equations (20), which is typical for most finite element analyses, determination of the amplification matrix eigenvalues is expensive. Fortunately, Irons & Treharne (1971) have shown that the system eigenvalues are bounded by the eigenvalues of individual elements. Thus, the eigenvalue evaluation of a single generic element suffices. From (19), the amplification matrix

$$\left(\mathbf{A} + \frac{2\,\Delta t}{3}\mathbf{B}\right)^{-1}\left(\mathbf{A} - \frac{\Delta t}{3}\mathbf{B}\right) \tag{33}$$

of the element has eigenvalues

$$\lambda_1 = 1 \qquad \lambda_2 = \frac{\rho c L_e^2 - 4k\Delta t}{\rho c L_e^2 + 8k\Delta t}. \tag{34}$$

Clearly, λ_1 fulfills the condition (32) and may be verified that λ_2 also fulfills the same condition for any time increment Δt.

According to Lax & Richtmyer (1956), the convergence of any discretization scheme is assured if it is consistent and stable. Consistency means that the discretized equation tends to the differential equation when mesh size tends to zero. In this sense, the developed element has a consistent formulation. As stability was proved by the eigenvalue bounds, the discretization scheme will also be convergent.

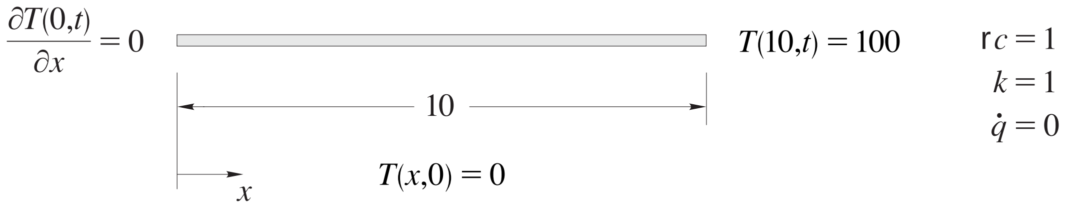

Figure 2: Bar under transient heat conduction

4 NUMERICAL EXAMPLE

The example to be considered is the transient heat conduction in a bar with $L = 10$, $\rho c = 1$, $k = 1$ and $\dot{q} = 0$ (consistent units are used) depicted in Figure 2. At the end $x = 0$ no heat flow is allowed and at the end $x = 10$ a temperature $T = 100$ is kept constant. Furthermore, the bar is initially at zero temperature. The following boundary conditions

$$\frac{\partial T(0,t)}{\partial x} = 0 \qquad T(10,t) = 100 \qquad t > 0 \tag{35}$$

and initial condition

$$T(x,0) = 0 \qquad 0 < x < 10 \tag{36}$$

are then prescribed.

The exact solution, given by

$$T(x,t) = 100 + \frac{400}{\pi}\sum_{i=1}^{\infty}\left[\frac{(-1)^i}{2i-1}\cos\frac{(2i-1)\,\pi x}{20}e^{-\frac{(2i-1)^2\,\pi^2 t}{400}}\right], \tag{37}$$

can be found in Al-Khoury (2012) by applying the Laplace transform to the heat conduction equation. It is easy to verify that the solution (37) satisfies the differential equation (1) at every point of the domain as well as the boundary conditions (35) exactly. However, the initial condition (36) is satisfied by the convergent series

$$T(x,0) = 100 + \frac{400}{\pi}\sum_{i=1}^{\infty}\left[\frac{(-1)^i}{2i-1}\cos\frac{(2i-1)\,\pi x}{20}\right]. \tag{38}$$

We say that the solution is exact, but not in closed form because the initial condition is satisfied by a series with infinite number of terms.

The numerical results were evaluated with 8 elements of equal size ($L_e = 1.25$) and 500 time increments of $\Delta t = 0.1$. Figure 3 shows results for $t = 0.1$ and $t = 0.4$ (that is, after one and four time increments). Figure 4 shows results for $t = 2$ and $t = 50$ (that is, after twenty and five hundred time increments). At the end $x = 10$, the exact solution is initially discontinuous which affects the numerical results by a visible temperature undershoot at the node adjacent to $x = 10$. The discontinuity effect upon the numerical results decreases quickly as t increases. A remarkable accuracy can be observed after a few time increments.

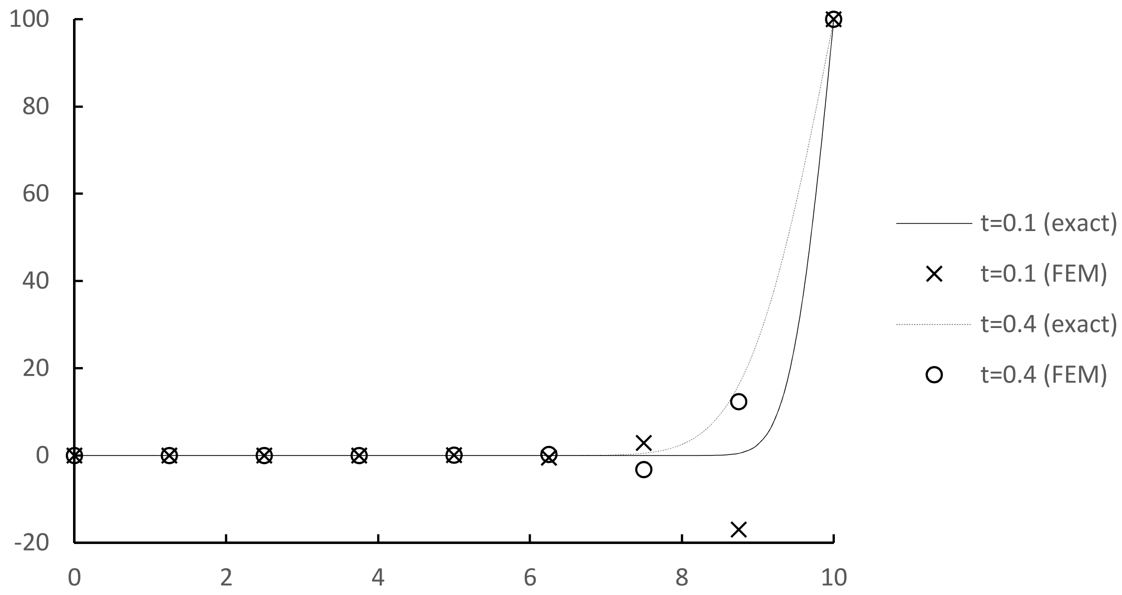

Figure 3: Numerical (FEM) and exact results for $t = 0.1$ and $t = 0.4$

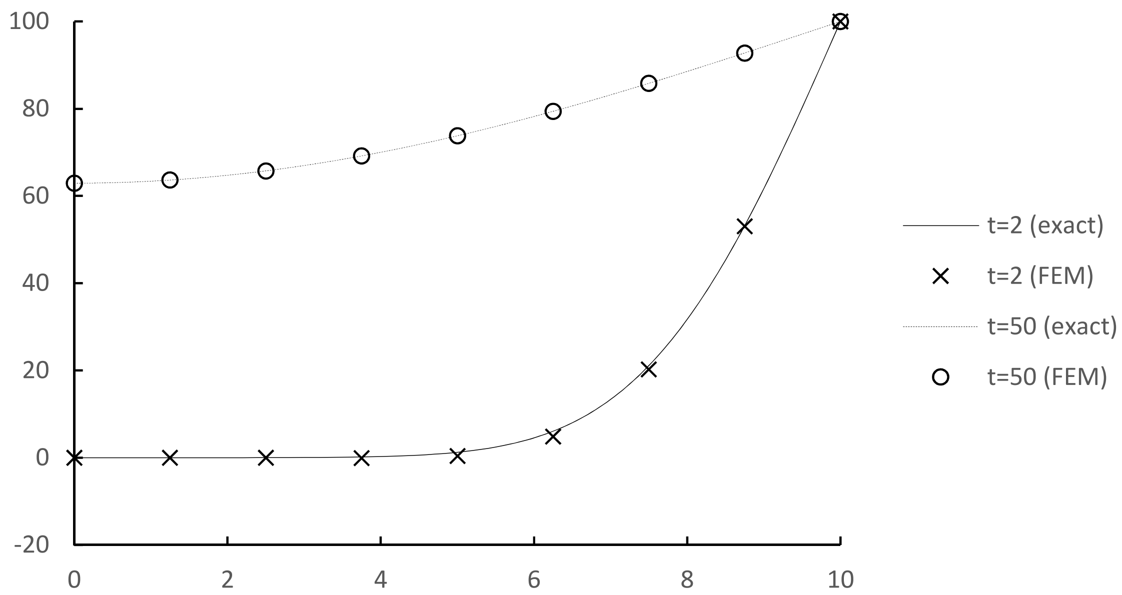

Figure 4: Numerical (FEM) and exact results for $t = 2$ and $t = 50$

5 CONCLUDING REMARKS

This paper presented an eigenvalue bound approach to assess the stability of a simple spatially one-dimensional finite element for transient heat conduction. The element is proved to be unconditionally stable, convergent and can lead to excelent results.

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