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BUCKLING OF PLATES SUBJECTED TO ROTATION-DEPENDENT LOADS

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Abstract. *This paper studies the buckling of thick plates subjected to loads which depend on the deformation of the structures, more specifically on the rotations about in-plane axes. In the cases studied in the paper the applied forces are not derivable from a potential function (nonconservative loading). The numerical model is a version of the Rayleigh-Ritz method, with appropriate global functions, with addition of nonsymmetrical matrices which statically simulate the load dependence on rotations. Special attention is given to the effect of shear deformation in the flutter load. It is confirmed that the dynamic buckling load is more affected by shear than the static buckling load, due to the modal interaction. A peculiar behavior is also observed in the case of an unusual loading which does not introduce primary in-plane forces in the plate. Examples of rectangular plats with different boundary conditions are presented.*

Keywords: *stability of plates, thick plates, nonconservative tangential follower loads, Rayleigh-Ritz method.*

1 INTRODUCTION

Classical problems of structural mechanics usually consider mechanical energy to be preserved. Usual loads are assumed to have a linear potential, which entails their invariance upon configuration changes. In structural analysis using finite elements, the nonlinear effects due to incipient buckling are usually introduced in the element matrices, generating the so-called geometric or initial stress matrices which allow the calculation of the overall buckling load for the structure. Forces with nonlinear potential, usually resulting from interaction with other structural systems, require modifications in the analysis which lead to additional effects and to significant changes in the buckling load level. The computation of critical loads in the case of a conservative system can be made in the realm of statics. However, if mechanical energy is not conserved, a dynamical approach is necessary, since unstable vibrations (flutter) may arise. Analysis of nonconservative systems is an important study associated with several problems in civil, mechanical, aerospace and naval engineering, and also in some fields of applied mechanics and engineering. Nonconservative forces commonly arise from the mechanical interaction of a continuous system with the environment surrounding it, such as wind pressure or impingent jets, or as a result of an external energy source.

Disregard of damping and more complex inertia effects in analytical or numerical models is cause of certain concern in the study of actual nonconservative forces, as mentioned by Elishakoff (2005). It should be remarked that the occurrence of certain flutter phenomena have been adequately predicted by models which represent the applied forces as “follower forces” or purely dependent on displacements and rotations (so-called circulatory forces). Early studies of nonconservative problems were presented by Bolotin (1963). Herrmann et al. (1966) provided interesting models of flutter under forces with follower characteristics.

More recently, Suanno and Silva (1990), presented interesting results for beams, highlighting the influence of shear deformation and rotatory inertia in the critical dynamic load (flutter load). Similar effects were demonstrated by Salas (2015), for circular plates.

The present study considers the buckling problem of plates, considering the thick theory for plates (Reissner-Mindlin) and also thin plate theory (Kirchhoff). For such special configurations, the Rayleigh-Ritz method with global functions is a feasible technique, as seen in Jarek (2007) e Salas (2015). The combination of internal functions and nodal functions is an interesting approach, allowing for a variety of different boundary conditions. A modified version of the energy method is required to include the follower loads which are non-conservative. This methodology was implemented for the Reissner-Mindlin theory. Two examples are compared, observing the effect of shear deformation and rotatory inertia in comparison with the same effect for static problems, the flutter load (Figure 1) and displacement dependent pressure load (Figure 2).

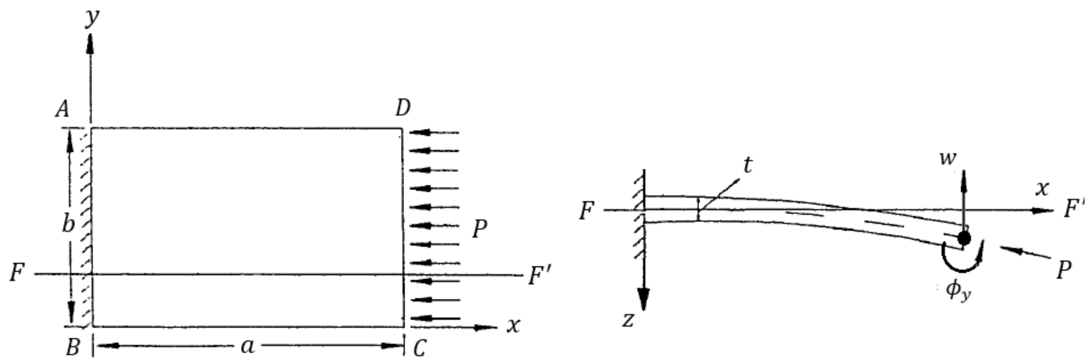


Figure 1. Geometry and dimensions of a plate in a cartesian coordinate system for flutter problem.

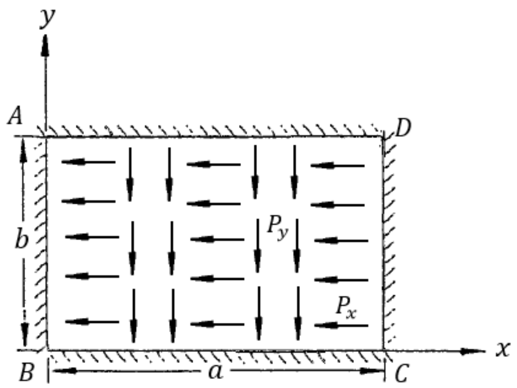


Figure 2. Geometry and dimensions of a plate in a cartesian coordinate system for the displacement dependent pressure load problem.

2 MATHEMATICAL FORMULATION

2.1 Description of the geometric parameters

Consider in Figure 1 a homogeneous and isotropic plate with thickness t. The origin of the coordinate system is taken at center of the plate in the mid-surface. Thus, the plate geometry and dimensions are defined by a cartesian coordinate system (x, y, z) .

2.2 The governing equation of motion

Based on the assumptions of Mindlin-Reisner plate theory, the displacement field may be described as follows:

$$\begin{aligned} u &= u_0 + z \cdot \theta_x \\ v &= v_0 + z \cdot \theta_y \\ w &= w_0 \end{aligned} \tag{1}$$

In the above equations, the terms u_0, v_0 and w_0 are the in-plane and transverse displacement components of the mid-surface of the plate; and θ_x and θ_y are in-plane rotations at the same points. In simple bending analysis, u_0 and v_0 may be set equal to zero.

2.3 Strain-displacement relations

For small displacements and strains, the strain-displacement relations may be described as follows:

$$\varepsilon_{xx} = z \cdot \frac{\partial \theta_x}{\partial x}; \quad \varepsilon_{yy} = -z \cdot \frac{\partial \theta_y}{\partial y}; \quad \varepsilon_{ZZ} = 0$$

$$\varepsilon_{xy} = -z \cdot \frac{\partial \theta_y}{\partial x} + z \cdot \frac{\partial \theta_x}{\partial y}; \quad \gamma_{xz} = z \cdot \frac{\partial \theta_x}{\partial x} + \frac{\partial w}{\partial x} \tag{2}$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} - z \cdot \frac{\partial \theta_y}{\partial y}$$

Where ε_{xx} and ε_{yy} are the normal strains and γ_{xy}, γ_{xz} *and* γ_{yz} are the engineering shear strains. This theory implies shear strains constant through the thickness, requiring the introduction of a shear correction factor.

2.4 Hooke's law

The stress-strain relations for the elastic plate can be written as

$$\sigma_{xx} = -\frac{E}{1-\upsilon^2} z \left(\frac{\partial \theta_x}{\partial x} + \upsilon \frac{\partial \theta_y}{\partial y}\right); \; \sigma_{yy} = -\frac{E}{1-\upsilon^2} z \left(\frac{\partial \theta_y}{\partial y} + \upsilon \frac{\partial \theta_x}{\partial x}\right); \sigma_{xy} = -\frac{E}{2(1+\upsilon)} z \left(\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}\right)$$

$$\sigma_{xz} = \frac{E}{2(1+\upsilon)} K \left(\theta_x + \frac{\partial w}{\partial x}\right); \; \sigma_{yz} = \frac{E}{2(1+\upsilon)} K \left(-\theta_y + \frac{\partial w}{\partial y}\right) \tag{3}$$

Where E and υ are the Young´s modulus and Poisson`s ratio, respectively; K denotes the transverse shear correction factor. It is known from the theory of beams that the shear stress varies parabolically by the thickness. According to Reddy (2007), the shear stress on plates varies at least quadratically through the thickness. This is usually corrected in the calculation of shear forces by introducing the shear correction factor K.

2.5 Rayleigh-Ritz method for analysis

The problem is to determine the buckling load of a thick plate under uniform compression in one direction. The load remains tangent to the mid-surface at the edge during the buckling process, which characterizes a nonconservative (path-dependent) behavior. The model for the plate contains nodal displacement functions, with three degrees of freedom per node in the contour and shape functions of an isoparametric quadrilateral in order to describe the displacement in the contour. In the model implemented the number of nodes can vary from 4 to 16 knots. Additional internal functions are hierarchical polynomials that do not correspond to any nodes, but improve the accuracy of the interpolation within the element. The accuracy of the Rayleigh-Ritz model is controlled by the number N of these internal functions. There is considerable freedom in the choice of the additional functions, except that they should be zero at the nodes.

The parameters defining the interpolation function in the plate are the translation (w) and two rotations (ϕ_x, ϕ_y) at points in the midsurface herein the approximation to be used for the displacement field uses a version of Legendre polynomials specially adapted for the problem.

$$N_{PL_\xi} = \frac{1}{2^{n_x} n_x!} \frac{d^{n_x}}{dx^{n_x}} [(\xi^2 - 1)^{n_x}] - C \tag{4}$$

In the above, C is a translation coefficient used to adapt the polynomial to the conditions of problem, varying according to the type of polynomial, odd or even (Abreu, 2015).

The boundary conditions are imposed by adding adequate spring stiffness values at the diagonals of the stiffness submatrix associated with the nodal degrees of freedom, zero representing a free displacement and a very large value (penalty approach) corresponding to a restraint. The non-nodal additional degrees of freedom are not affected.

This approach satisfies exactly the essential boundary conditions (displacements and rotations), and is convergent for other boundary conditions.

2.6 Equation of motion

For free vibration, the kinetic energy *T* and the strain energy *U* of an elastic thick plate are expressed as

$$T = \frac{1}{2} \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \int_{-t/2}^{t/2} \rho(\dot{u}^2 + \dot{v}^2 + \dot{w}^2)\, dxdydz \tag{5}$$

$$U = \frac{1}{2} \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \int_{-t/2}^{t/2} \left(\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{xy}\gamma_{xy} + \sigma_{xz}\gamma_{xz} + \sigma_{yz}\gamma_{yz}\right) dxdydz \tag{6}$$

The potential energy corresponding to the conservative component *V* is given by

$$V = \frac{1}{2} \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \int_{-t/2}^{t/2} \left(\sigma_{xx}{\theta_x}^2 + \sigma_{yy}{\theta_y}^2 + \sigma_{xy}\theta_x\theta_y\right) dxdydz \tag{7}$$

In the above expressions, ρ is the specific mass of the plate material. σ_{xx} and σ_{yy} are the stresses depend on the uniform compressive load *P* distributed along the edge.

2.7 Eigenvalue problem

For linear small-strain assuming harmonic motion, the global displacement vectors can be written as

$$x = \bar{x}e^{i\omega t} \tag{9}$$

$$\bar{x} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{10}$$

Here $\bar{x}$ is a nonzero vector of constants to be determined, ω is the natural frequency. Substituting Equations (5), (6) and (7) into the expressions (8) leads to the governing eigenvalue equation of the form

$$(\bar{K}_E + \lambda_{CR}(\bar{K}_G + \bar{K}_L + \bar{K}_{NL}) - \omega^2 \bar{M})\bar{x} = 0 \qquad (11)$$

In the above expression, $\bar{K}_E$ is the stiffness matrix, $\bar{K}_G$ is the geometric matrix, $\bar{K}_L$ is the symmetric load matrix, $\bar{K}_{NL}$ is the nonsymmetric load matrix, $\bar{M}$ is the mass matrix, λ_{CR} is the load parameter and ω is the frequency. Clearly, matrix $\bar{K}_{NL}$ cannot be obtained from the energy functional, but results from consideration of equilibrium in the deformed configuration.

3 RESULTS AND DISCUSSIONS

For the following examples, it consider a plate with $a = 5.0m$, $b = 5.0m$, $E = 210GPa$, $\nu = 0.25$ and $\rho = 8.0\ 10^{-8} KN/cm^3$. The methodology was first applied in a static example, defining the first three natural frequencies and static critical load, compared to the finite element model (SAP2000) to observe the convergence and effectiveness of internal functions. In order to describe the displacement in the contour shape functions of an isoparametric quadrilateral with 12 nodes have been used.

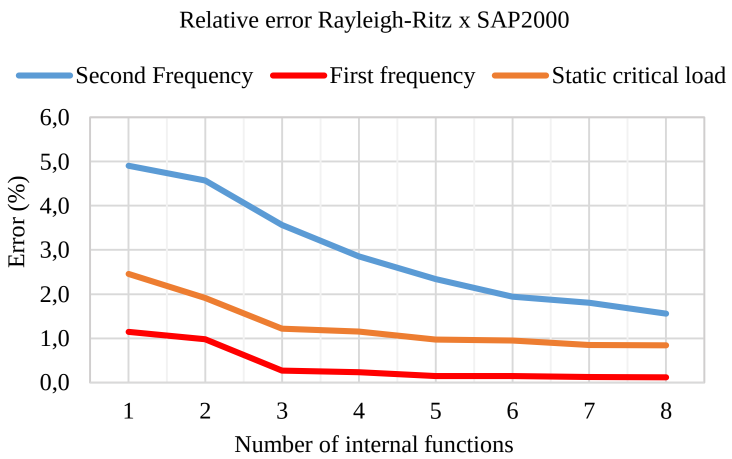


Figura 3. Relative error between the models of internal functions and finite elements for the cantilever plate example.

The first three vibration mode shapes of the cantilever plate are presented in Fig. 4. These first two modes are most influential in the analysis of flutter under non-conservative tangential follower loads.

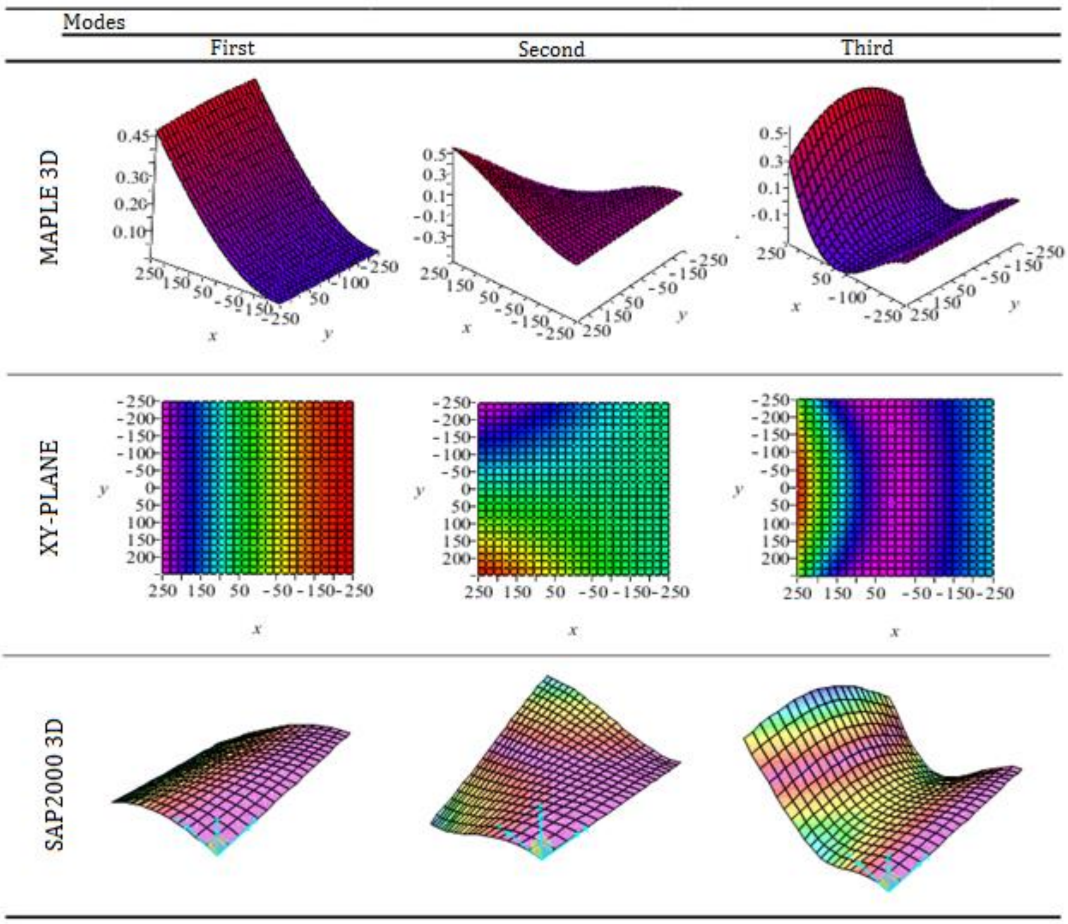


Figure 4. Deformed mode shape of a cantilever plate.

For the non-conservative tangential follower loads problem, the variation of the first two frequencies of plates subjected to uniform axial load of increasing magnitude is depicted in Figs. 5. The load is compressive but remains tangential to the mid-surface during buckling (follower load), hence the problem is nonconservative. Contrary to the static analysis, where the lowest frequency decreases to zero at buckling load, here the first frequency increases with increasing load. Flutter occurs when the first two frequencies coalesce, just prior to become complex conjugate pairs. The presence of an imaginary positive part implies a vibration of increasing amplitude in time.

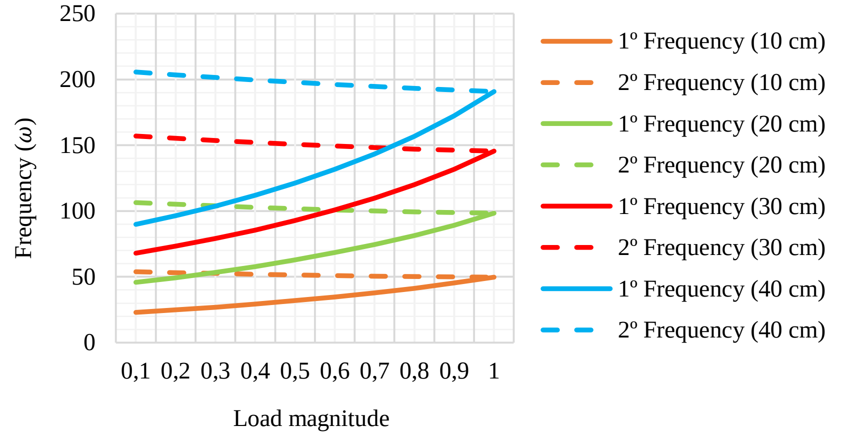


Figure 5. Coalescence of the first two frequencies of a cantilever plate.

In Fig. 5 a plates are considered according to thick plate theory (shear deformation and rotatory inertia included). The first frequency is particularly affected as the load magnitude are varied from 0,1 to 1, because of the higher shear deformation content in this mode. Hence, the coalescence and consequently the flutter load vary more pronouncedly, with respect to the static buckling load, with this ratio varying from approximately 8.20 to 7.31.

For comparison and observation of the effects considered, an equivalent model was developed using a well-known finite element analysis software (SAP2000) with the thin plate theory (Kirchkoff).

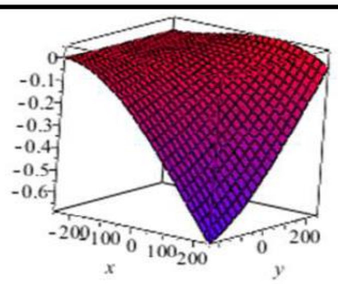
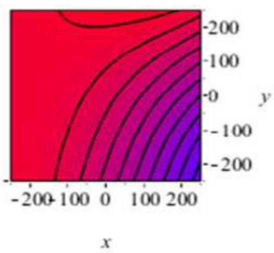
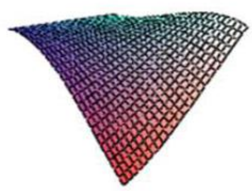
Mode		
MAPLE 3D	XY - PLANE	SAP2000 3D
		

Figure 6. Vibration mode resultant of the flutter load

The coalescence of the first two frequencies is reached when the structure is subjected to a load magnitude equal to approximately 8.33 times the magnitude of the static buckling load. Very little change in the load of coalescence (flutter load) is observed, relative to the static buckling load, it should be remarked that such small changes should disappear when rotatory inertia effects are removed.

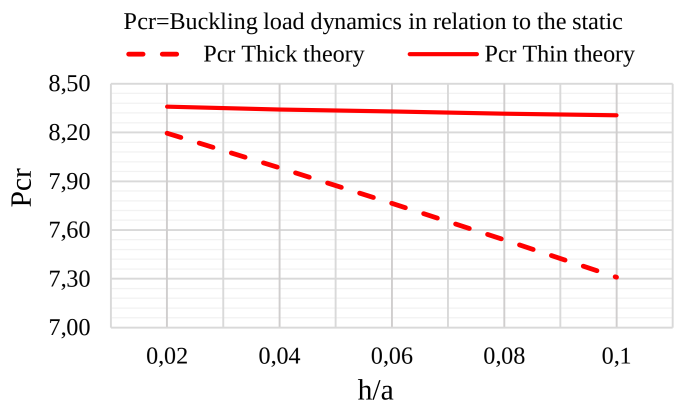


Figure 7. Variation of the flutter load

Such effects are further illustrated in Fig. 7 where the continuous line indicates the results obtained according to thin plate theory, while the dashed line corresponds to the results considering shear deformation and rotational inertia (thick plate theory). The vertical axis correspond to the flutter load divided by the static critical load, in each theory, and the horizontal axis correspond to the ratio thickness/length (h/a). The drop in the relative load value is more accentuated for the thick plate (approximately 8.20 to 7.31) than for the thin plate (8.24 to 8.21). Again, this is mostly due to the fact that the first is significantly affected by shear deformation. In the case of thin plates, the variation is due only to rotatory inertia, usually disregarded. It is interesting that such results resemble the findings for beams in bending, though here the problem is further complicated by the possibility of modes with torsional content in addition to primary bending.

The follower forces in the classical Beck's problem (Bolotin, 1963), similarly to the above example, lead to a flutter buckling load several times higher than the static buckling load. In certain exceptional cases, nonconservative forces could lead to static buckling (divergence) at a load level below the case of dead loads, as described by Naudascher and Rockwell (1994).

In this paper, we present in the second example an unusual loading, in which there is the effect of uniform in-plane distributed load which remains tangent to the midsurface. It is found that such loading induces static buckling only. However, if the initial membrane stress in the plate is eliminated, there is no geometric stiffness matrix, but there is a load stiffness matrix. This is clearly an even more difficult loading to realize in practice, which makes it liable to criticism (in the line of comments by Elishakoff, 2005). However, such difficulty is considered herein a challenge for further research in the field since we speculate that certain flexible structures are susceptible to collapse in this fashion. In this peculiar situation, criticality was found to occur by flutter only. The implementation of this type of loading in standard commercial finite element packages was troublesome, reason why we do not comparative results.

The results are illustrated on Figs. 8 and 9, where it is seen the appearance of coalescence of the two first frequencies in a way similar to the first example. Further studies are being made in order to identify actual situations where such load behavior would occur.

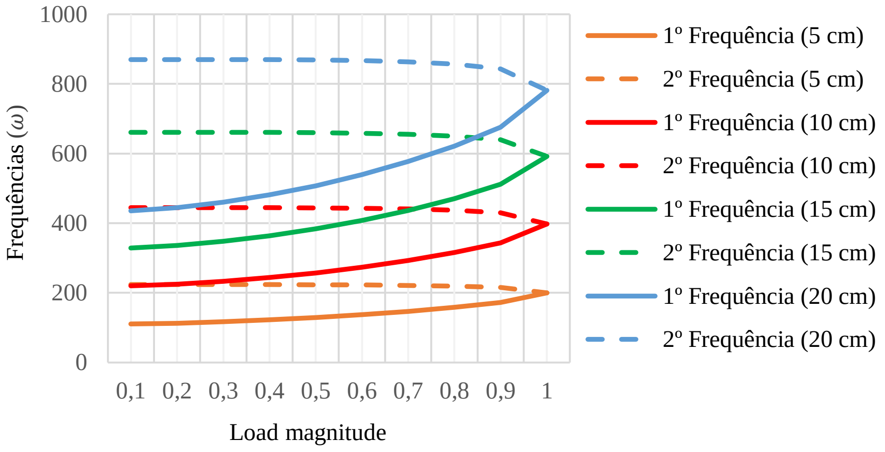


Figure 8. Coalescence of the first two frequencies of a clamped plate with displacement dependent pressure load.

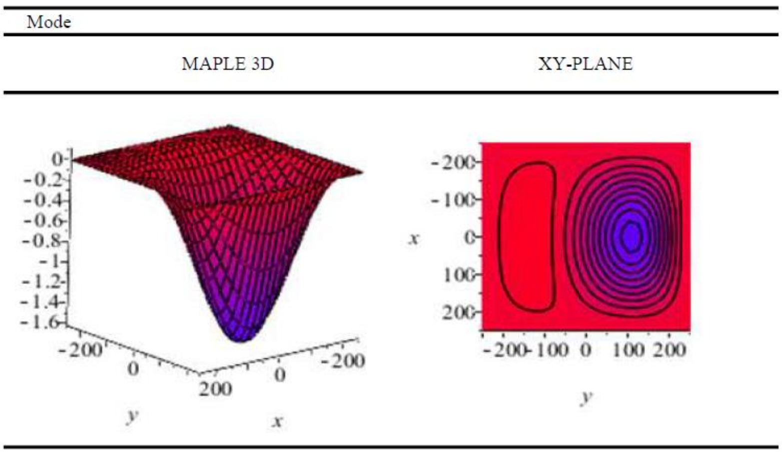


Figure 9. Vibration mode resultant of the flutter load for a clamped plate with displacement dependent pressure load

4 CONCLUSIONS

The paper presented two example of nonconservative load applications, the flutter problem of a cantilever plate under uniform axial compressive load acting along the outer edge and remaining tangent to the mid-surface when buckling occurs and displacement dependent pressure load applied on surface.

Concerning the method implemented herein, the combination of conventional and additional functions in a Rayleigh-Ritz type of approximation, it should be mentioned that no orthogonalization was implemented for this set of functions, which precludes the use of a large number of functions in a similar way to the mesh refinement in conventional finite elements. A drawback of the method is to lead to full matrices and to larger computer times in conventional solvers.

The dependence of results on the thickness is explained (Fig. 7) by the effect of shear deformation, which differs for the first and second modes of vibration involved in the flutter load, causing a pronounced change in the point of coalescence of the frequencies. In the case of the thin plate, the relation between the flutter load and the static buckling load depends slightly on the thicknesses/length, due to the effect of rotatory inertia.

In the case of the second example, with tangent loading, we found that the critical condition correspond to divergence (static buckling). However, if the load does not induce initial membrane stress in the plate (a loading whose possibility is to be discussed in future studies), it is remarkable that only flutter occurs.

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