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BUCKLING OF ANISOTROPIC PLATES BY THE RITZ METHOD

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Abstract. *The Ritz method is used in the buckling analysis of anisotropic plates under several combinations of in-plane loads and boundary conditions. Ritz bases are generated from modified Legendre polynomials, and the plate rigidities are carefully chosen to provide thermodynamically admissible materials. The accuracy of the proposed approach is assessed by means of several examples solved by finite element models.*

Keywords: *Ritz method, Anisotropic plates, Buckling analysis*

1 INTRODUCTION

For certain edge and loading conditions, the exact solution for the linear buckling problem of isotropic or orthotropic plates can be assessed by means of classical approaches, such as the Lévy method (Reddy, 2004). For anisotropic plates, however, the presence of bending-twisting coupling significantly increases the complexity of the analysis which requires approximate solutions. High gradients of the buckling mode and nonsatisfaction of some boundary conditions in a strict pointwise sense has brought numerical difficulties to the analysis of anisotropic plates using powerful tools like the Ritz method: a larger number of terms in the approximating function is required to obtain accurate solutions than is required for similar accuracy when dealing with isotropic or orthotropic plates.

The Ritz method can pose low-grade convergence or numerical instability depending on the function basis selected, no matter whether anisotropy is involved or not. For instance, the use of beam functions over constrains plates with free edges (Bassily and Dickinson, 1975; Dickinson and Di Blasio, 1986), and high-order polynomials can trigger numerical instabilities for being too much alike (Hjelmstad, 2005) or evaluated with round-off errors (Beslin and Nicolas, 1997). Anisotropic plates with simply supported edges are also over-constrained by beam functions (Nallim and Grossi, 2003; Gawandi et al., 2008).

Herein an extension of our previous formulation (Monteiro et al., 2014) is presented where the plate is now anisotropic. The Ritz bases are generated from modified Legendre polynomials proposed by Zhu (1986) (see also Bardell (1991)) and the plate rigidities are carefully chosen to provide thermodynamically admissible materials. All results are obtained by a discretization scheme that treats each plate as it were a single finite element, with refinement carried out by adding hierarchic modes of higher order. The accuracy of the proposed approach is assessed by means of several examples solved by finite element models.

2 RITZ EQUATIONS

The homogeneous anisotropic rectangular plate shown in Fig. 1, of length a and width b, is subjected to uniform in-plane loads p_x, p_y, p_{xy} and has the midsurface in the xy-plane of the Cartesian coordinate system xyz. According to the Kirchhoff theory, the plate buckling is described by

$$
\begin{aligned}
&D_{11} w_{,xxxx} + 2\left(D_{12} + 2D_{66}\right) w_{,xxyy} + D_{22} w_{,yyyy} + 4D_{16} w_{,xxxy} + 4D_{26} w_{,xyyy} \\
&+ p_x w_{,xx} + 2p_{xy} w_{,xy} + p_y w_{,yy} = 0
\end{aligned}
\tag{1}
$$

where w is the displacement in the z direction, subscripts x and y preceded by commas denote differentiation with respect to x or y, and D_{ij} are the bending stiffnesses of the plate. A laminated plate composed of layers that are symmetrically disposed, both from material and geometric properties standpoint, about the midsurface has also the buckling described by (1) for which D_{ij} depend now on the material properties, layer thicknesses and the lamination scheme (Reddy, 2004; Whitney, 1987).

The solution of Eq (1) should satisfy prescribed values of

$$
\begin{aligned}
&w \text{ or } V_x \qquad w_{,x} \text{ or } M_x \qquad \text{on} \quad x = 0,\, x = a \\
&w \text{ or } V_y \qquad w_{,y} \text{ or } M_y \qquad \text{on} \quad y = 0,\, y = b
\end{aligned}
\tag{2}
$$

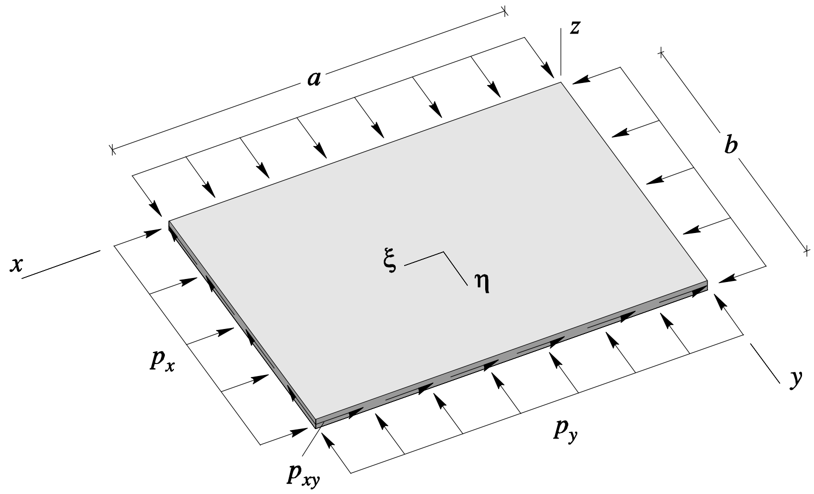

Figure 1: Rectangular anisotropic plate subjected to uniform in-plane loads p_x, p_y, p_{xy}

where V_x, V_y are the effective shear forces and M_x, M_y are the bending moments. One element of each pair (w, V_x), (w, V_y), $(w_{,x}, M_x)$ and $(w_{,y}, M_y)$ (but not both elements of the same pair) may be specified. Moreover, the displacement w or the twisting moment M_{xy} (but not both) should be specified at the plate corners. A boundary condition is called geometric when w, $w_{,x}$ or $w_{,y}$ are specified and is called mechanical when V_x, V_y, M_x, M_x or $M_{,xy}$ are specified.

Equation (1) and homogeneous form of the mechanical boundary conditions ($V_x = 0$ or $M_x = 0$ on $x = 0$, $x = a$; $V_y = 0$ or $M_y = 0$ on $y = 0$, $y = b$; $M_{xy} = 0$ at the plate corners) can be stated as the stationary condition $\delta\Pi = 0$ of the potential energy

$$\Pi = \frac{1}{2}\int_0^b \int_0^a \left(D_{11}w_{,xx}^2 + 2D_{12}w_{,xx}w_{,yy} + D_{22}w_{,yy}^2 + 4D_{66}w_{,xy}^2 \right. \\ \left. +4\left(D_{16}w_{,xx} + D_{26}w_{,yy}\right)w_{,xy} - p_x w_{,x}^2 - 2p_{xy}w_{,x}w_{,y} - p_y w_{,y}^2\right) dx\, dy \quad (3)$$

with respect to w (Washizu, 1982).

It is convenient to nondimensionalize the potential energy by adopting the coordinates $\xi = (2x - a)/a$ and $\eta = (2y - b)/b$ to obtain

$$\Pi = \frac{2}{ab}\int_{-1}^1 \int_{-1}^1 \left(D_{11}\frac{b^2}{a^2}w_{,\xi\xi}^2 + 2D_{12}w_{,\xi\xi}w_{,\eta\eta} + D_{22}\frac{a^2}{b^2}w_{,\eta\eta}^2 + 4D_{66}w_{,\xi\eta}^2 \right. \\ \left. +4\left(D_{16}\frac{b}{a}w_{,\xi\xi} + D_{26}\frac{a}{b}w_{,\eta\eta}\right)w_{,\xi\eta} - p_x\frac{b^2}{4}w_{,\xi}^2 - p_{xy}\frac{ab}{2}w_{,\xi}w_{,\eta} - p_y\frac{a^2}{4}w_{,\eta}^2\right) d\xi\, d\eta. \quad (4)$$

The solution of the buckling problem is approximately sought by the Ritz method in the form

$$w(\xi, \eta) \approx \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}F_i(\xi)F_j(\eta), \quad (5)$$

whose coefficients c_{ij} are determined from $\delta\Pi = 0$. The approximation Eq. (5) should satisfy the geometric boundary conditions.

Substitution of Eq. (5) into Eq. (4) leads to

$$\sum_{k=1}^{m}\sum_{l=1}^{n}\sum_{i=1}^{m}\sum_{j=1}^{n}\Bigg(D_{11}\frac{b^2}{a^2}A_{ki}^{22}B_{lj}^{00}+D_{12}\left(A_{ki}^{20}B_{lj}^{02}+A_{ki}^{02}B_{lj}^{20}\right)+D_{22}\frac{a^2}{b^2}A_{ki}^{00}B_{lj}^{22}$$
$$+4D_{66}A_{ki}^{11}B_{lj}^{11}+2D_{16}\frac{b}{a}\left(A_{ki}^{21}B_{lj}^{01}+A_{ki}^{12}B_{lj}^{10}\right)+2D_{26}\frac{a}{b}\left(A_{ki}^{01}B_{lj}^{21}+A_{ki}^{10}B_{lj}^{12}\right)$$
$$-p_x\frac{b^2}{4}A_{ki}^{11}B_{lj}^{00}-p_{xy}\frac{ab}{4}\left(A_{ki}^{10}B_{lj}^{01}+A_{ki}^{01}B_{lj}^{10}\right)-p_y\frac{a^2}{4}A_{ki}^{00}B_{lj}^{11}\Bigg)c_{ij}\,\delta c_{kl}=0 \tag{6}$$

where

$$A_{ki}^{rs}=\int_{-1}^{1}\frac{d^rF_k}{d\xi^r}\frac{d^sF_i}{d\xi^s}d\xi \qquad B_{lj}^{rs}=\int_{-1}^{1}\frac{d^rF_l}{d\eta^r}\frac{d^sF_j}{d\eta^s}d\eta. \tag{7}$$

Since δc_{kl} are arbitrary and independent, the discretized version of the buckling problem is given by

$$\Bigg[D_{11}\frac{b^2}{a^2}\left[\Delta^{2200}\right]+D_{12}\left(\left[\Delta^{2002}\right]+\left[\Delta^{0220}\right]\right)+D_{22}\frac{a^2}{b^2}\left[\Delta^{0022}\right]$$
$$+4D_{66}\left[\Delta^{1111}\right]+2D_{16}\frac{b}{a}\left(\left[\Delta^{2101}\right]+\left[\Delta^{1210}\right]\right)+2D_{26}\frac{a}{b}\left(\left[\Delta^{0121}\right]+\left[\Delta^{1012}\right]\right)$$
$$-p_x\frac{b^2}{4}\left[\Delta^{1100}\right]-p_{xy}\frac{ab}{4}\left(\left[\Delta^{1001}\right]+\left[\Delta^{0110}\right]\right)-p_y\frac{a^2}{4}\left[\Delta^{0011}\right]\Bigg]\{c\}=\{0\} \tag{8}$$

with

$$\left[\Delta^{pqrs}\right]=\begin{bmatrix}\left[\Delta_{11}^{pqrs}\right] & \dots & \left[\Delta_{1m}^{pqrs}\right]\\ \vdots & & \vdots\\ \left[\Delta_{m1}^{pqrs}\right] & \dots & \left[\Delta_{mm}^{pqrs}\right]\end{bmatrix} \qquad \left[\Delta_{ki}^{pqrs}\right]=A_{ki}^{pq}\begin{bmatrix}B_{11}^{rs} & \dots & B_{1n}^{rs}\\ \vdots & & \vdots\\ B_{n1}^{rs} & \dots & B_{nn}^{rs}\end{bmatrix}$$
$$\{c\}=\lfloor\,\lfloor\, c_{11}\ \cdots\ c_{1n}\,\rfloor\ \cdots\ \lfloor\, c_{m1}\ \cdots\ c_{mn}\,\rfloor\,\rfloor^T. \tag{9}$$

Equation (8) can be written simbolically as the linear eigenvalue problem

$$\left(\left[K_c\right]-\lambda\left[K_g\right]\right)\{c\}=\{0\}\,, \tag{10}$$

with the constitutive stiffness matrix

$$\left[K_c\right]=D_{11}\frac{b^2}{a^2}\left[\Delta^{2200}\right]+D_{12}\left(\left[\Delta^{2002}\right]+\left[\Delta^{0220}\right]\right)+D_{22}\frac{a^2}{b^2}\left[\Delta^{0022}\right]+4D_{66}\left[\Delta^{1111}\right]$$
$$+2D_{16}\frac{b}{a}\left(\left[\Delta^{2101}\right]+\left[\Delta^{1210}\right]\right)+2D_{26}\frac{a}{b}\left(\left[\Delta^{0121}\right]+\left[\Delta^{1012}\right]\right) \tag{11}$$

and the geometric stiffness matrix $[K_g]$ expressed according to the loading type. For instance,

$$
\begin{aligned}
[K_g] &= \frac{b^2}{4}\left[\Delta^{1100}\right] && \lambda = p_x \ (p_y = p_{xy} = 0)\\
[K_g] &= \frac{b^2}{4}\left[\Delta^{1100}\right] + \frac{a^2}{4}\left[\Delta^{0011}\right] && \lambda = p_x = p_y \ (p_{xy} = 0)\\
[K_g] &= \frac{b^2}{4}\left[\Delta^{1100}\right] - \frac{a^2}{4}\left[\Delta^{0011}\right] && \lambda = p_x = -p_y \ (p_{xy} = 0)\\
[K_g] &= \frac{ab}{4}\left(\left[\Delta^{1001}\right] + \left[\Delta^{0110}\right]\right) && \lambda = p_{xy} \ (p_x = p_y = 0).
\end{aligned}
\tag{12}
$$

3 NUMERICAL RESULTS

The mode functions $F_i(\xi)$ and $F_j(\eta)$ in Eq. (5) are taken as the hierarchical polynomial set

$$
\begin{aligned}
F_1(\xi) &= \frac{1}{2} - \frac{3}{4}\xi + \frac{1}{4}\xi^3 \qquad F_2(\xi) = \frac{1}{8} - \frac{1}{8}\xi - \frac{1}{8}\xi^2 + \frac{1}{8}\xi^3\\
F_3(\xi) &= \frac{1}{2} + \frac{3}{4}\xi - \frac{1}{4}\xi^3 \qquad F_4(\xi) = -\frac{1}{8} - \frac{1}{8}\xi + \frac{1}{8}\xi^2 + \frac{1}{8}\xi^3\\
F_i(\xi) &= \sum_{n=0}^{(i-1)/2} \frac{(-1)^n}{2^n n!}\frac{(2i-2n-7)!!}{(i-2n-1)!}\xi^{i-2n-1} \qquad i = 5, 6, \ldots
\end{aligned}
\tag{13}
$$

where $i!! = i(i-2)\ldots(2 \text{ or } 1)$, $0!! = (-1)!! = 1$, $(i-1)/2$ denotes its own integer part. The first four modes are identical to the Hermite cubic polynomials and account for the geometric boundary conditions (displacement and rotation at $\xi = \pm 1$), while the higher order modes ($i = 5, 6, \ldots$) have been generated from Legendre polynomials by Zhu (1986) (see also Bardell (1991)) and possess both zero displacement and zero slope at each end. This features are significant since the higher modes contribute only to the internal displacement field, and all the classical geometric boundary conditions could be matched just by removing some of the first four basis functions. In the same manner as is done on finite element procedures, the boundary conditions may be introduced a posteriori into Eq. (10).

We have written a code in MATLAB language to solve the linear eigenvalue problem given by Eq. (10). Integrals in Eq. (7) are evaluated by symbolic computing to circumvent round-off errors and matrix $[\Delta^{pqrs}]$ is generated and stored in advance for $m = n = 104$ in Eq. (5). The latter procedure, which reduces computation cost in the evaluation of $[K_c]$ and $[K_g]$ for any plate modeled with $m, n \leq 104$, is only possible because: (a) $[\Delta^{pqrs}]$ is independent of plate geometry and material; (b) we know that the geometric boundary conditions are accounted by the first four functions $F_i(\xi)$ and $F_j(\eta)$. In order to run the following examples with $m = n = 20$ and introduce the geometric boundary conditions we have simply removed exceeding lines and columns from the previously stored $[\Delta^{pqrs}]$.

3.1 Example 1

A number of tests are conducted using several combinations of loading and boundary conditions in order to verify the buckling behavior of anisotropic square plates subject to one of the following uniform loading: axial compression p_x ($p_y = p_{xy} = 0$), biaxial compression $p_x = p_y$ ($p_{xy} = 0$), compression p_x and tension $p_y = -p_x$ ($p_{xy} = 0$), shear p_{xy} ($p_x = p_x = 0$). For purpose of description, the boundary conditions at the plate edges are denoted by: SSSS (all the edges are simply supported), SSCC (edges are simply supported at $x = 0, a$ and clamped at $y = 0, b$), SSCF (edges are simply supported at $x = 0, a$, clamped at $y = 0$ and free at $y = b$), CCCC (all the edges are clamped).

The adopted bending stiffnesses, with consistent units, are

$$D_{11} = D_{22} = 1 \quad D_{12} = 0.03 \quad D_{66} = 0.735 \quad D_{16} = D_{26} = -0.5. \tag{14}$$

Such parameters make the stiffness matrix

$$\begin{bmatrix} D_{11} & D_{12} & D_{16} \\ & D_{22} & D_{26} \\ \text{sym.} & & D_{66} \end{bmatrix} \tag{15}$$

positive definite, as expected for a thermodynamically admissible material. A suitable measure of the anisotropy, given by

$$\frac{D_{16}}{\sqrt[4]{D_{11}^3 D_{22}}} = \frac{D_{26}}{\sqrt[4]{D_{11} D_{22}^3}} = -0.5 \tag{16}$$

according to Weaver and Nemeth (2007), indicate that the plates are moderately anisotropic.

Table 1 shows that the results from Ritz solutions are in excellent agreement with those obtained from a fine mesh of Nastran CQUAD4 elements (Nx Nastran, 2014) for any loading and boundary conditions. The differences between the results from both procedures are not greater than 0.6%.

3.2 Example 2

Suppose the simply supported square plate subjected to axial compression p_x ($p_y = p_{xy} = 0$) shown in Fig. 2 is composed of a single unidirectional lamina of P100/AS3501 prepreg material, with properties

$$E_1 = 369 \text{ GPa} \quad E_2 = 5.03 \text{ GPa} \quad G_{12} = 5.24 \text{ GPa} \quad \nu_{12} = 0.31. \tag{17}$$

Young's moduli E_1 and E_2 refer to x_1 and x_2 directions, shear modulus G_{12} and Poisson's ratio ν_{12} refer to x_1x_2 plane. Table 2 shows how the plate anisotropy varies with the lamina rotation. For $\theta = 0°$ and $\theta = 90°$ the plate is orthotropic, and around $\theta = 45°$ it reaches high anisotropy since $D_{16}/\sqrt[4]{D_{11}^3 D_{22}}$ and $D_{26}/\sqrt[4]{D_{11} D_{22}^3}$ are bounded by ± 1 for simply supported plates (Weaver and Nemeth, 2007). No matter how severe the anisotropy is, the results from Ritz and finite element solutions are in excellent agreement with differences not greater than 0.7%. As expected, the maximum difference occurs for $\theta = 40°$ and $\theta = 45°$ when anisotropy is most severe.

Table 1: Buckling load for square plates

	$p_x a^2/\pi^2 D_{22}$ $(p_{xy}=0)$			$p_{xy} a^2/\pi^2 D_{22}$
Type	$p_y = 0$	$p_y = p_x$	$p_y = -p_x$	$(p_x = p_y = 0)$
SSSS	3.302	1.653	7.089	3.722
	0.998†	0.998	0.998	0.995
SSCC	5.908	3.067	9.293	5.235
	0.998	0.998	1.000	0.994
SSCF	1.574	1.144	2.297	2.466
	0.999	0.998	0.999	0.997
CCCC	7.615	4.028	12.524	6.124
	0.998	0.998	0.998	0.994

† Ritz/Nastran

4 CONCLUSIONS

A Ritz scheme based on a set of modified Legendre functions has been presented for buckling analysis of anisotropic plates under several combinations of in-plane loads and boundary conditions. Very accurate and stable solutions have been obtained for all cases considered, including plates with severe anisotropy, with reduced computational cost.

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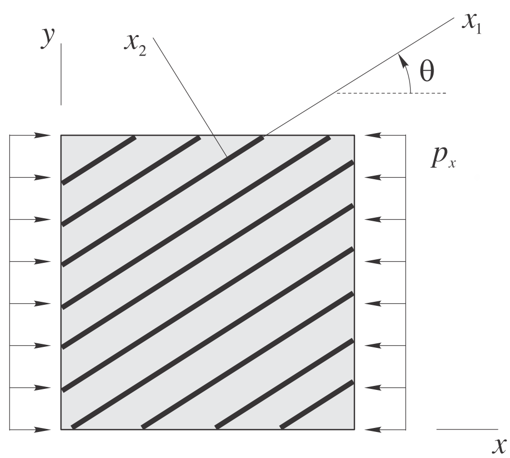

Figure 2: Simply supported square plate composed of a single unidirectional lamina

Table 2: Buckling load for SSSS square plates with several θ

θ	$D_{16}/\sqrt[4]{D_{11}^3 D_{22}}$	$D_{26}/\sqrt[4]{D_{11} D_{22}^3}$	Ritz	Ritz/Nastran
0°	0.000	0.000	9.240	1.000
10°	0.475	0.187	8.400	1.000
20°	0.785	0.599	5.401	1.000
30°	0.893	0.828	2.994	1.002
40°	0.917	0.902	2.085	1.007
45°	0.914	0.914	1.921	1.007
50°	0.902	0.917	1.845	0.999
60°	0.828	0.893	1.859	1.000
70°	0.599	0.785	2.328	0.999
80°	0.187	0.475	2.660	1.000
90°	0.000	0.000	2.561	1.000

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