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SEMI-ANALYTICAL APPROACH FOR ELASTOPLASTIC ANALYSIS OF STEEL BEAMS

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Abstract. *The stress and displacement analysis is necessary to design and build efficiently any structure. In general, the current accepted procedures to design steel beams admit the linear-elastic material behavior and are based on the assumptions adopted by the Euler-Bernoulli beam theory. However, it is critical to predict the behavior of the structural elements under plastic deformation, in order to understand the failure modes and the plastic hinges formation in steel beams. Accordingly, this paper proposes a theoretical procedure to evaluate stresses and displacements in steel beams under elastoplastic deformation, assuming the kinematic assumptions of the Euler-Bernoulli beam theory. Moreover, an iterative-incremental procedure based on the reformulated Prandtl-Reuss equations proposed by Mendelson is employed to compute the plastic strains, in the context of a secant or total elastoplastic formulation. In order to verify the proposed approach, its results are compared with those ones obtained by the finite element method analysis. This comparison demonstrates the effectiveness of the proposed model, and shows the possibility to realize more realistic and sophisticated analysis of steel beams using this approach.*

Key-words: *Steel beams, Euler-Bernoulli beam theory, Elastoplastic analysis, Theory of plasticity, Reformulated Prandtl-Reuss equations.*

1 INTRODUCTION

The majority of the current procedures for designing structures assume that the applied external load will lead to a linear-elastic material behavior, which means that strains are admitted to be reversible and the linear stress-strain relationship is valid. However, the last decades have produced significant improvements in the analysis and understanding of structural behavior of bodies under plastic deformation (see, for instance, Baker and Heyman (1969); Horne (1979); Moy (1981); Neal (1977)). In addition, Baker (1949) has shown that the most economical projects of continuous beams were achieved when the design made use of the plastic range.

In fact, in the civil construction field, engineers have to design structures in the linear-elastic range to avoid excessive deformations, but in others kind of structures, for instance some parts of vehicles, the elastoplastic range has to be put into account to increase the capacity of the element to absorb the strain energy. Along those lines, it becomes clear that the understanding of structural behavior of bodies under elastoplastic deformation plays an important role for structural engineering.

Accordingly, Mendelson (1968) points out that the theory of plasticity falls into two categories: physical theories and mathematical theories. The former intends to explain why some solid bodies flow plastically, looking microscopically to the material to understand what happens to the atoms, crystals and grains when plastic flows occurs. The later, on the other hand, are systematic and provide one with the necessary tools to formulate and solve engineering problems. Therefore, this paper focus on the mathematical theories of plastic flows while proposes a systematic semi-analytical approach for elastoplastic analysis of steel beams.

Provided that, the proposed model assumes the kinematic assumptions of the Euler-Bernoulli beam theory in the formulation process and makes use of an interactive-incremental procedure based on the reformulated Prandtl-Reuss equations proposed by Mendelson to compute the plastic strains. Consequently, this approach enables one to realize more realistic and sophisticated analysis of steel beams as its numerical results agree with those ones obtained by finite element method analysis.

2 SEMI-ANALYTICAL APPROACH FOR ELASTOPLASTIC ANALYSIS OF BEAMS

As follows, this paper describes the method employed to achieve the proposed model. Firstly, an analytical formulation to evaluate stress components has been developed and derived assuming the kinematic assumptions of the Euler-Bernoulli beam theory. Secondly, for plastic strain evaluation, this work made use of an iterative-incremental procedure based on the reformulated Prandtl-Reuss equations proposed by Mendelson. Finally, three numerical methods were applied: the central finite difference method to approximate derivatives, the trapezoidal method for numerical integrations and the improved Euler method to evaluate the deflections. Those numerical methods will be further explained in section 2.3.

2.1 Formulation of Stress Components

In this section, this work derives a formulation for the moment-curvature relationship, the normal stresses (σ_{xx}) and the shear stresses (τ) for an elastoplastic deformation. First of all, for the case of a member subjected to a bending moment applied perpendicularly to its longitudinal axis, the kinematic assumptions of the Euler-Bernoulli beam theory are the following:

i. The longitudinal axis (x), which lies within the neutral surface, does not experience any change in length.

ii. All plane cross sections remain plane and orthogonal to the deformed axis.

iii. Any deformation of the cross section within its own plane will be neglected.

In other words, the axial, torsion and shear effects are neglected and, thus, the bending moment is the critical internal force. With this in mind, it can be verified that, for elastic strain states, the neutral axis is the same of the horizontal centroidal axis, however for elastoplastic strain states, it will not be always true. Figure 1 illustrates the difference between the strain diagrams for those aforementioned strain states and it also shows the variable $\bar{y}$, which represents the distance from the horizontal centroidal axis to the neutral surface.

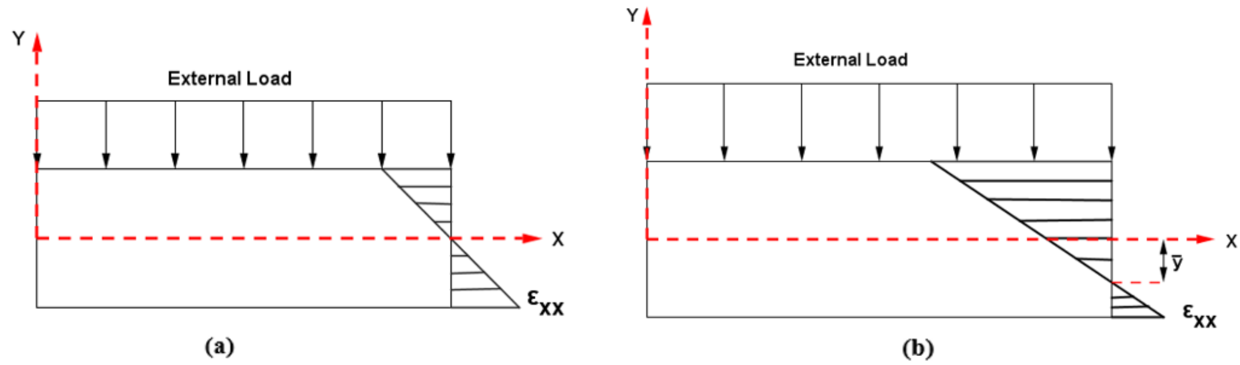


Figure 1: (a) – Strain diagram for an elastic strain state; (b) – Strain diagram for an elastoplastic strain state.

Besides, Eq. (1) represents the total strain (ε_{xx}) and it comes straightly from assumption (i). In addition, from the additive decomposition principle of the strain, ε_{xx} can be also expressed as shown in Eq. (2).

$$\varepsilon_{xx} = -K(x)(y - \bar{y}) \quad (1)$$

$$\varepsilon_{xx} = ({\varepsilon_{xx}}^{e} + {\varepsilon_{xx}}^{p}) \quad (2)$$

Where $K(x)$ represents the curvature, ${\varepsilon_{xx}}^{e}$ represents the elastic strain and ${\varepsilon_{xx}}^{p}$ corresponds to the plastic strain. Satisfying the condition that the resultant force produced by the stress distribution over the cross-sectional area must be equal to zero and combining Eq. (1) and Eq. (2), $\bar{y}$ can be derived as follows:

$$F_x = \int_A \sigma_{xx}\, dA = 0 \rightarrow \int_A [-K(x)(y - \bar{y}) - {\varepsilon_{xx}}^{p}]E\, dA = 0 \quad (3)$$

Solving this equation for $\bar{y}$ and making the adequate operations:

$$\bar{y} = \frac{\overline{{\varepsilon_{xx}}^{p}}}{k(x)} \quad (4)$$

Where $\overline{{\varepsilon_{xx}}^{p}} = \frac{1}{A}\int_A {\varepsilon_{xx}}^{p} dA$. Now, it is possible to develop a formula for the moment-curvature relationship.

$$M(x) = \int_A -y\sigma_{xx}\, dA \;\rightarrow\; M(x) = E[K(x)\int_A y^2\, dA - K(x)\bar{y}\int_A y\, dA + \int_A y\,{\varepsilon_{xx}}^p\, dA \tag{5}$$

Solving Eq. (5) for $K(x)$, replacing $\bar{y}$ with Eq. (4) and making the adequate algebraic simplifications, $K(x)$ can be written as:

$$K(x) = \frac{M(x)}{EI} - \frac{1}{I}\int_A y\,{\varepsilon_{xx}}^p\, dA \tag{6}$$

Here, $M(x)$ is the resultant internal moment, E represents the modulus of elasticity of the material and I corresponds to the moment of inertia of the cross-sectional area about the neutral axis.

Similarly, the flexure formula for an elastoplastic strain state is derived using the additive decomposition principle of the strain and the moment-curvature relationship.

$$\sigma_{xx}(x,y) = E(\varepsilon_{xx} - {\varepsilon_{xx}}^p) \;\rightarrow$$

$$\sigma_{xx}(x,y) = -\frac{M(x)}{EI}(y-\bar{y}) + \frac{E}{I}(y-\bar{y})\int_A y\,{\varepsilon_{xx}}^p\, dA - E{\varepsilon_{xx}}^p \tag{7}$$

Finally, the transverse shear formula will be developed considering the horizontal force equilibrium of an element as show in Fig. 2. Hence, applying the equation of horizontal force equilibrium and using Eq. (7):

$$\sum F_x = 0 \rightarrow \int_{A'} \sigma' dA' - \int_{A'} \sigma\, dA' - \tau(x,y)(tdx) = 0 \;\rightarrow$$

$$\int_{A'}\left\{-\frac{M(x+\Delta x)}{I}[y-\bar{y}(x+\Delta x)] + \frac{E}{I}[y-\bar{y}(x+\Delta x)]\int_A y\,{\varepsilon_{xx}}^p(x+\Delta x,y)\, dA - E{\varepsilon_{xx}}^p(x+\Delta x,y)\right\} dA'$$
$$-\int_{A'}\left\{-\frac{M(x)}{I}[y-\bar{y}(x)] + \frac{E}{I}[y-\bar{y}(x)]\int_A y\,{\varepsilon_{xx}}^p(x,y)\, dA - E{\varepsilon_{xx}}^p(x,y)\right\} dA' = \tau(x,y)t\Delta x \tag{8}$$

After an exhaustive algebraic manipulation and making Δx tending to zero, the transverse shear formula can be written as:

$$\tau(x,y) = \left[-Q(x) + E\int_A y\,\frac{\partial {\varepsilon_{xx}}^p}{\partial x} dA\right]\frac{\bar{y}' - \bar{y}}{It}A' + \left[M(x) - E\int_A y\,{\varepsilon_{xx}}^p\, dA\right]\frac{1}{It}\frac{d\bar{y}}{dx}A' - \frac{E}{t}\int_{A'}\frac{\partial {\varepsilon_{xx}}^p}{\partial x} dA' \tag{9}$$

Where $\bar{y}' = \frac{1}{A'}\int_{A'} y\, dA'$.

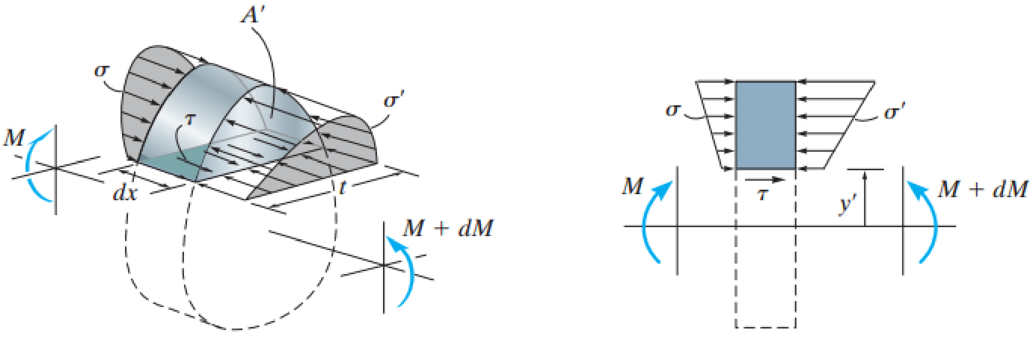


Figure 2: Three-dimensional and profile view. Source: Hibeller (2011).

2.2 Plastic Strain Evaluation

This section is based on Mendelson (1968) which proposes an iterative-incremental procedure to compute the plastic strain increments from the total strains without resource to the stresses. Firstly, assume a general external load path to a given state of stress and total plastic strains ${\varepsilon_{ij}}^p$. Now, an increment of plastic strain $(d{\varepsilon_{ij}}^p)$ is produced by an increment of load, so the total strain can be written as:

$$\varepsilon_{ij} = {\varepsilon_{ij}}^e + {\varepsilon_{ij}}^p + d{\varepsilon_{ij}}^p \tag{10}$$

Where ${\varepsilon_{ij}}^e$ is the elastic component of the total strain, ${\varepsilon_{ij}}^p$ is the accumulated plastic strain up to the current increment of load, $d{\varepsilon_{ij}}^p$ is the increment of plastic strain due to the increment of load and the equation is written in indicial notation. Furthermore, Eq. (11) defines the modified total strains as follows:

$$\varepsilon'_{ij} = \varepsilon_{ij} - {\varepsilon_{ij}}^p \tag{11}$$

Then, nothing that $\varepsilon'_{ij} = {\varepsilon_{ij}}^e + d{\varepsilon_{ij}}^p$ and subtracting the mean strain from the diagonal components of both sides results in:

$$e'_{ij} = {e_{ij}}^e + d{\varepsilon_{ij}}^p \tag{12}$$

Where ${e_{ij}}^e$ is the elastic strain deviator tensor and e'_{ij} is the modified strain deviator tensor. From Hooke's law and the Prandtl-Reuss equations:

$$e'_{ij} = \left(1 + \frac{1}{2Gd\lambda}\right) d{\varepsilon_{ij}}^p \tag{13}$$

Where G is the shear modulus and $d\lambda$ is a nonnegative constant which may vary throughout the loading history. Now, defining an equivalent modified total strain by $e'_e = \sqrt{\frac{2}{3} e'_{ij} e'_{ij}}$ and making the appropriated manipulations:

$$d{\varepsilon_{ij}}^p = \frac{d{\varepsilon_e}^p}{e'_e} e'_{ij} \tag{14}$$

Where $d{\varepsilon_e}^p = \sqrt{\frac{2}{3} d{\varepsilon_{ij}}^p d{\varepsilon_{ij}}^p}$ is the effective plastic strain increments. Equation (14) is equivalent to the Prandtl-Reuss equations, however the stresses do not appear in it and the increments of plastic strain can be computed from the total strains. In addition, since Eq. (14) have been derived by use of the Prandtl-Reuss equations, it implicitly makes use of the von Mises yield criterion. Finally, $d{\varepsilon_e}^p$ can be computed making use of the following equation:

$$d{\varepsilon_{e(i)}}^p = \frac{e'_e - \frac{\sigma_{e(i-1)}}{3G}}{1 + \frac{\frac{d\sigma_e}{d{\varepsilon_e}^p}_{(i-1)}}{3G}} \tag{15}$$

Where $\sigma_{e\,(i-1)} = \sigma_y + \frac{EH}{E-H} {\varepsilon_e}^p{}_{(i-1)}$ considering a plastic with linear hardening material behavior. Here, H represents the plastic modulus and ${\varepsilon_e}^p = \int d{\varepsilon_e}^p$.

2.3 Numerical Methods

In order to compute the derivative and integrates on the proposed model, two numerical methods were employed; they are, respectively: central finite difference and trapezoidal methods. In addition, to evaluate the deflection of the beam, the improved Euler method was used. Accordingly, this paper briefly presents these three numerical methods.

Firstly, the central finite difference method is one of the simplest finite difference approximation of the derivative. It approximates the derivative of a function at a point ('x=a') using values of the function at different points in the neighborhood of 'x=a'. This method estimates the derivative from values of two points by the value of the slope of the line that connects the two points (Amos and Subramaniam (2014)). Furthermore, the central difference is computed using Eq. (16).

$$\frac{df}{dx}(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{x_{i+1} - x_{i-1}} \tag{16}$$

Secondly, the trapezoidal method is based on a refinement over simple rectangle and midpoint methods and it makes use of Newton's form of interpolating polynomials with two points ('x=a' and 'x=b'). Succinctly, the numerical integration is made with Eq. (17).

$$I(f) = \frac{[f(a) + f(b)]}{2}\,(b - a) \tag{17}$$

Lastly, the improved Euler method consists in a discretization of the beam into *n* points $(x_0, x_1, \dots, x_i, x_{i+1}, \dots, x_{n-1}, x_n)$ and then the inclination θ_{i+1} is determined from the previous inclination (θ_i) (Noronha and Cavalcante (2015)). In this process the starting point is a previously known boundary condition (θ_0). Equation (18) is used to evaluate the inclinations:

$$\theta_{i+1} \approx \theta_i + (x_{i+1} - x_i)\frac{\kappa_i + \kappa_{i+1}}{2} \tag{18}$$

Likewise, the deflections can be computed from the inclination values as follow:

$$v_{i+1} \approx v_i + (x_{i+1} - x_i)\frac{\theta_i + \theta_{i+1}}{2} \qquad (18)$$

Finally, it is necessary to emphasize that the deflections can only be estimated because the curvature (κ_i) was firstly computed with Eq. (6).

3 NUMERICAL RESULTS

The proposed approach was implemented using MatLab® and the numerical results are shown in this section. To verify the effectiveness of the method, those numerical results were compared with those ones obtained using Abaqus®, which is a software that applies finite element methods. Along these lines, the numerical results include: the effective plastic strain field, the horizontal stresses field, the transverse shear stresses field, deflections and others parameters that will not be shown in this paper due to restrictions on its "length". In addition, it is important to mention that two beams configuration were analyzed: a cantilever beam and a simply supported beam.

3.1 Cantilever Beam

Material properties: ASTM A36 steel was employed in the analysis and it has an Elasticity Modulus (*E*) of 200 GPa, Poisson's Ratio (*v*) of 0.26 and Yield Stress (σ_y) of 250 MPa. Additionally, it was assumed a Plastic Modulus (*H*) of 20 GPa.

Dimensions of the beam: the beam has a length (*L*) of 1m, width (*b*) of 0.05m and height (*h*) of 0.1m.

Load pattern: a concentrated vertical load (P) was, transversely, applied at the free end of the beam and it has a magnitude of 50 kN.

After delineating the aforementioned parameters, the analysis could be performed. Figure 3 evidences the deflection, the effective plastic strain field, the horizontal stresses field and the transverse shear stresses field along the length of the beam. The effective plastic strains values are dimensionless and the stresses values are in Pa.

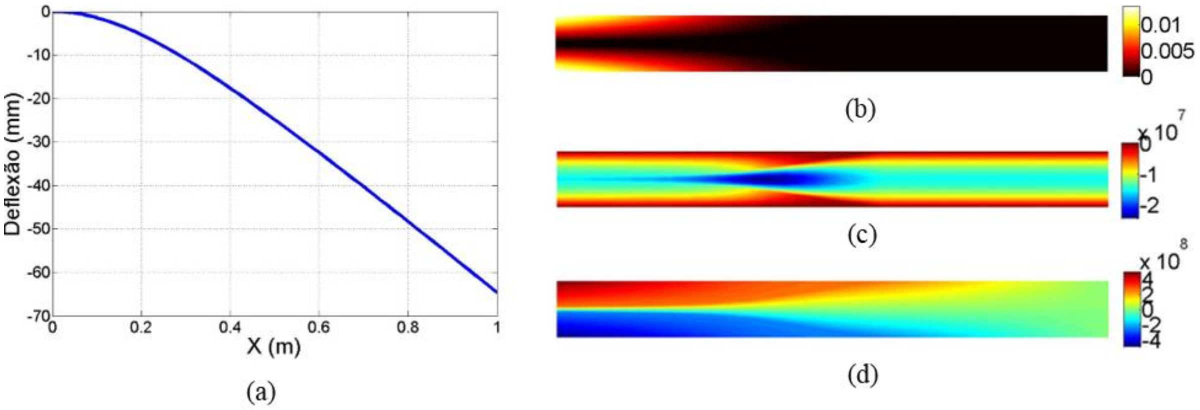


Figure 3: Cantiler beam. (a) deflection of the beam; (b) effective plastic strain field; (c) transverse shear stresses field; (d) horizontal stresses field.

Figure 3(a) represents the deflection of the beam, where can be observed a deflection of approximately 65mm at its free end. In the first sight, it may seem exaggerated, but it is in accordance with the proportion of the load. Moreover, according to Fig. 3(b) plastic hinges formation starts in the fixed end, and increasingly propagates with the increment of load. This effective plastic strain pattern was expected since stresses in the fixed end are expressively great and they decrease with the distance from the fixed end. Similarly, the transverse shear and horizontal stresses fields patterns (Fig. 3(c) and Fig. 3(d)) are physically consistent and expected since it is in agreement with Eq. (9) and Eq. (7), respectively.

3.2 Simply Supported Beam

Material properties: the material used in the simply supported beam analysis was exactly the same of that one described in section 3.1.

Dimensions of the beam: the beam has a length (L) of 3m, width (b) of 0.15m and height (h) of 0.3m.

Load pattern: a distributed vertical load (P) was, transversely, applied at the upper side of the beam and it has a magnitude of 650 kN/m.

Figure 4 represents the deflection, the effective plastic strain field, the horizontal stresses field and the transverse shear stresses field along the length of the beam. The effective plastic strains values are dimensionless and the stresses values are in Pa.

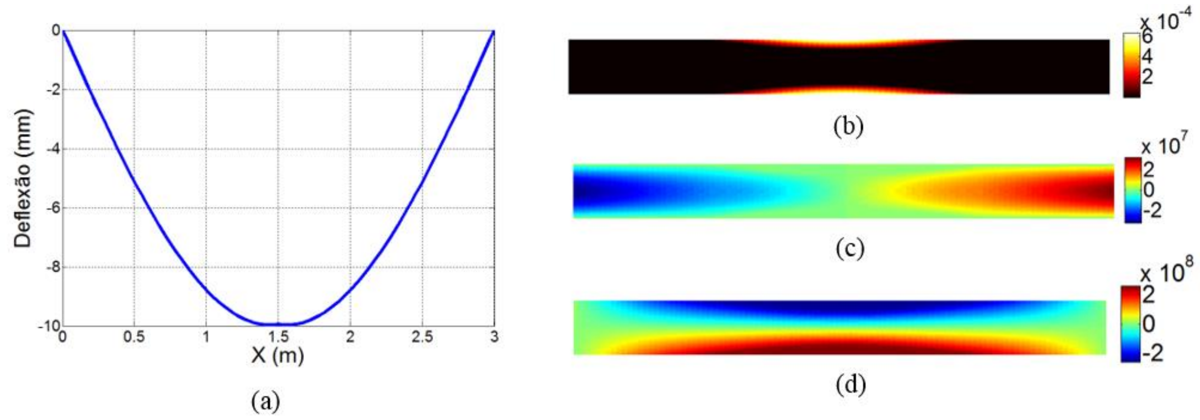
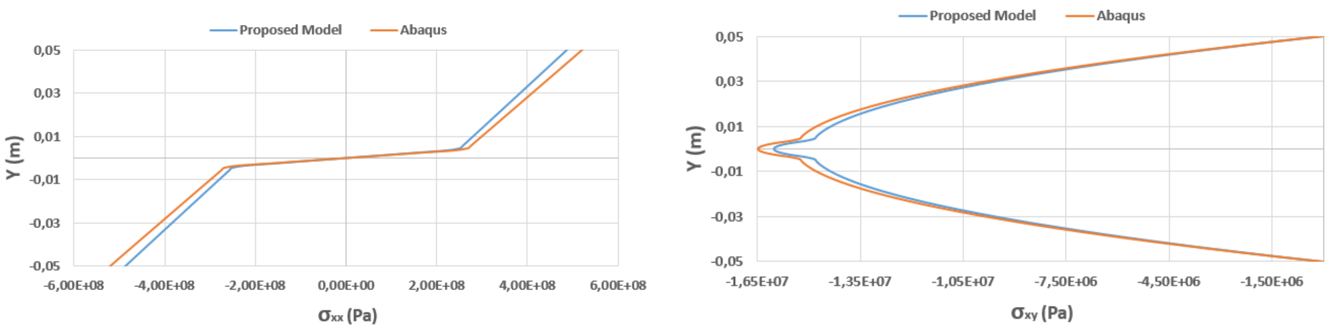


Figure 4: Simply supported beam. (a) deflection of the beam; (b) effective plastic strain field; (c) transverse shear stresses field; (d) horizontal stresses field.

Figure 4(a) represents the deflection of the beam and its maximum value, which is 10mm, lies in the mid-span of the beam as expected in order to agree with Eq. (6). Furthermore, the effective plastic strains (Fig. 4(b)) are greater in the mid-span region of the beam as expected, since the internal bending moment reach its maximum value at this region. Likewise, the transverse shear and horizontal stresses fields patterns (Fig. 4(c) and Fig. 4(d)) are physically consistent and expected since it is in agreement with Eq. (9) and Eq. (7), respectively.

3.3 Results Comparison and Discussions

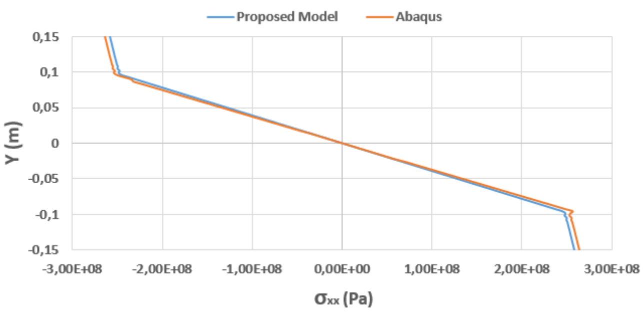
In this section, the results are compared with values obtained by the finite element method analysis, in order to verify the proposed approach. Figure 5 illustrates a comparison of those two methods for the cantilever beam's case.



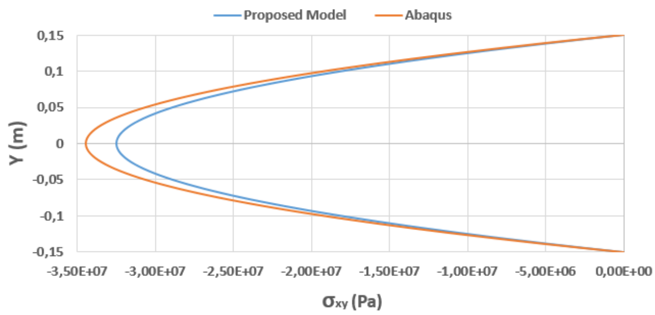
(a) (b)

Figure 5: (a) horizontal stress diagram at the fixed end section; (b) transverse shear stress diagram at the fixed end section.

Noticeably, the proposed model's results are very close to those obtained with Abaqus® for both horizontal and transverse shear stress diagrams, which testify the effectiveness of the proposed model. In addition, Fig. 6 shows a comparison of those two methods for the simply supported beam's case.



(a)



(b)

Figure 6: (a) horizontal stress diagram at the mid-span section; (b) transverse shear stress diagram at the end section.

Again, the proposed model's results are very close to those obtained with Abaqus® for both horizontal and transverse shear stress diagrams. As can be noticed in Fig. 5 and Fig. 6 the results are not perfectly equal for the proposed approach and the finite element method, but their divergences were expected since the proposed model derived makes use of some assumptions outlined on section 2.1 while the finite element method is a more accurate procedure.

Besides, the transverse shear stress formula derivation (Eq. 9) made use of some assumptions and a very critical one is that the transverse shear stresses are constant throughout the width of the beam. Obviously, this assumption plays an important role in the results differences between the methods.

Finally, the maximum deflections obtained by the proposed approach, for the cantilever and simply supported beam's case, were 65 mm and 10 mm, respectively. Likewise, the maximum deflections obtained by the finite element method, for the cantilever and simply supported beam's case, were 68 mm and 10.3 mm, respectively. Those values are considerably close.

Therefore, the proposed approach provides a very good level of accuracy and effectiveness since the results are noticeably close to the results obtained with Abaqus®.

4 CONCLUSIONS

This paper proposes an innovative approach for elastoplastic analysis of steel beams. In general, the current accepted procedures to design steel beams admit the linear-elastic material behavior and are based on the assumptions adopted by the Euler-Bernoulli beam theory. However, it is critical to predict the behavior of the structural elements under plastic deformation, in order to understand the failure modes and the plastic hinges formation in steel beams.

To fill this gap, this work developed a semi-analytical approach to compute stresses, strains and displacements in steel beams under elastoplastic deformation. To do so, it was assumed the kinematic assumptions of the Euler-Bernoulli beam theory and an iterative-incremental procedure based on the reformulated Prandtl-Reuss equations proposed by Mendelson was employed to compute the plastic strains. This approach is certainly useful since it provides values for deflections, effective plastic strains, transverse shear stresses and horizontal stresses fields, beyond it gives imagistic resources which contribute to the celerity of the interpretation.

This approach, was verified with a comparison of the results obtained with the proposed model and those ones from Abaqus®. This comparison demonstrates the effectiveness of the proposed model, and shows the possibility to realize more realistic and sophisticated analysis of steel beams using this approach. Besides, this method showed to not only be effective but also to be easy to understand and reproduce.

Finally, this paper might contribute to the expansion of scientific investigations of the elastoplastic deformations phenomena and serve as a source for engineer and scientist around the world.

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