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Nitsche's Method - An approach to imposing essential boundary conditions in Element Free Galerkin Implemented in INSANE

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Abstract. *Mesh-free methods use nodes to establish a system of algebraic equations. One of the advantages of mesh free methods is their independency of element connectivity, allowing some freedom in dealing with complex problems, such as large deformation, crack propagation, complex geometry, fluid flow, among others. The Element Free Galerkin is an example of such methods. As some mesh-free methods, its shape functions do not present the Kronecker Delta property, which is one of the reasons that the imposition of essential boundary conditions is not trivial as it is in FEM, for instance. There is a large effort to finding an efficient strategy for imposition of essential boundary conditions in mesh-free methods, besides the well known Lagrange multipliers, penalty and FEM coupling methods. As an alternative, Nitsche's method presents a consistent variational formulation and renders a better conditioned system matrix as it requires a smaller scalar factor to be used, in comparison to the penalty method. It also maintains the size of the original algebraic system of equations as opposed to the Lagrange multiplier method. However, the generalization and implementation of this method is not straightforward and is problem dependent in contrast to the methods aforementioned. The aim of this paper is to show the results of an implementation of the Nitsche's method in INSANE and compare the results of different methods for imposition of essential boundary conditions against it.*

Keywords: *Mesh-free, Boundary Conditions, Insane, Nitsche's Method*

1 INTRODUCTION

Mesh Free Methods are attractive to represent phenomena that have some interface problems, as different materials, fluid analysis and elaborated geometry. These methods do not require an initial point interconnection, a mesh, between the nodes. This is one of the reasons such methods are used in elaborated geometry problems. However, the shape functions of these methods, usually, do not have the Delta Kronecker property. Thus, the imposition of essential boundary conditions is not straightforward, as it is compared to the Finite Element Method. The aim of this work is to show the results of an implementation in INSANE (INteractive Structural ANalysis Environment) of the Nitsche's method for imposition of essential boundary conditions. This paper presents the Element Free Galerkin, a meshless method implemented in this software. Following a section about the boundary conditions methods used in this paper to compare the results with the ones obtained using Nitsche's method. Then it's presented a numerical example. Finally, a conclusion is drawn.

2 ELEMENT FREE GALERKIN

The Element Free Galerkin (EFG) (Belytschko et al. (1994)) is based on the Diffuse Element Method of Nayroles et al. (1992), and it is based in a functional interpolation of the form given a number of particles, x_j in the domain, Ω, as written in Eq. 1.

$$u(\mathbf{x}) \simeq u_p(\mathbf{x}) = \sum u(\mathbf{x}_j) N_j(\mathbf{x}) \tag{1}$$

where

$$N_j(\mathbf{x}) = \sum_{i=1}^{\ell} p_i(\mathbf{x}) \left[\mathbf{M}^{-1}(\mathbf{x})\, \mathbf{B}(\mathbf{x})\right]_{ij} = \mathbf{p}^T \mathbf{M}^{-1} \mathbf{B}_j, \tag{2}$$

recalling that ℓ is the number of terms of the polynomial basis $\mathbf{p}(\mathbf{x})$, N_j the EFG shape functions, $\mathbf{M}$ the moment matrix given by Eq. 3, $\mathbf{B}_j = \phi_j(\mathbf{x})\, \mathbf{p}(\mathbf{x}_j)$ is an auxiliary variable, and $\phi(\cdot)$ is the weighting function.

$$\mathbf{M}(\mathbf{x}) = \sum_{j}^{n} \phi_j(\mathbf{x})\, \mathbf{p}(\mathbf{x}_j)\, \mathbf{p}^T(\mathbf{x}_j) \tag{3}$$

The discrete system of equations is based on the Galerkin Weak Form of the Eq. 4. It can be understood as a meshless method, however it is necessary a set of background cells to perform the numerical integration of the discrete system, Liu (2009).

The partial different equations of a solid mechanics problem can be written:

$$\begin{aligned} \mathbf{L}^T\boldsymbol{\sigma} + \mathbf{b} &= \mathbf{0} \quad \text{Equilibrium in the domain } \Omega \\ \mathbf{u} &= \mathbf{u}_d \quad \text{on the essential boundary } \Gamma_d \\ \mathbf{n}^T\boldsymbol{\sigma} &= \mathbf{t} \quad \text{on the natural boundary } \Gamma_n \end{aligned} \tag{4}$$

where $\mathbf{L}$ is the differential operator, $\boldsymbol{\sigma}$ is the stress vector, $\mathbf{u}$ is the displacement vector, $\mathbf{b}$ is the body force vector, $\mathbf{t}$ is a vector of the prescribed force in the boundary, $\mathbf{u}_d$ is the prescribed displacement on the essential boundary, and finally, $\mathbf{n}$ is the unit normal outward on the boundary.

3 BOUNDARY CONDITIONS

As it was mentioned earlier, it is not so trivial, as it is in the Finite Element Method, to imposing the essential boundary conditions. There are some techniques which modify the shape functions, as coupling with finite elements, Silva (2012), or those that modify the weak Galerkin form, as the penalty and the Lagrange multipliers methods. Besides these methods there is the Nitsche's method, which is similar to the penalty method, but employs a smaller scalar, rendering a better conditioned system of equations. In the following, the weak form of problem (4) is depicted for each of the later methods to imposing the essential boundary conditions.

- **Penalty Method**: This method uses a scalar parameter, β, to imposing the essential boundary condition. This parameter is usually large, and it can lead to ill conditioned system matrices.

$$\left(\int_{\Omega} \mathbf{B}^T\mathbf{D}\mathbf{B}\, d\Omega + \beta \int_{\Gamma_d} \mathbf{N}^T\mathbf{N}\, d\Gamma\right)\mathbf{u} = \int_{\Omega} \mathbf{N}^T\mathbf{b}\, d\Omega + \int_{\Gamma_n} \mathbf{N}^T\mathbf{t}\, d\Gamma + \beta \int_{\Gamma_d} \mathbf{N}^T\mathbf{u}_d\, d\Gamma \tag{5}$$

  where $\mathbf{B}$ is the strain matrix, $\mathbf{D}$ the constitutive matrix, $\mathbf{N}$ are the EFG shape functions and $\mathbf{u}$ is the unknown MLS parameters vector.

- **Lagrange Multipliers**: The Lagrange multiplier method is well known and largely used in a variety of problems. However, it implies in adding more variables in the system of equations, which renders to a semi-positive definite system matrix.

$$\left(\int_{\Omega} \mathbf{B}^T\mathbf{D}\mathbf{B}\, d\Omega - \int_{\Gamma_d} \mathbf{\Phi}^T\mathbf{N}\, d\Gamma\right)\mathbf{u} = \int_{\Omega} \mathbf{N}^T\mathbf{b}\, d\Omega + \int_{\Gamma_n} \mathbf{N}^T\mathbf{t}\, d\Gamma \tag{6}$$
$$\left(\int_{\Gamma_d} \mathbf{N}^T\mathbf{\Phi}\, d\Gamma\right)\boldsymbol{\lambda} = -\int_{\Gamma_d} \mathbf{\Phi}^T\mathbf{u}_d\, d\Gamma$$

  Here, $\mathbf{\Phi}$ are the Lagrange multipliers shape functions and $\boldsymbol{\lambda}$ is the Lagrange multipliers unknown vector.

- **Nitsche's Method**: The Nitsche's method is similar to the penalty method since it also uses a scalar β, however not as large as the one used in penalty method, which leads to a better conditioned system matrix as mentioned by Fernández-Méndez and Huerta (2004).

  According to Huerta et al. (2004) this method maintains the consistency of the weak form of problem (4). For futher details see Hah et al. (2014) and Embar et al. (2010).

$$\begin{aligned}\left(\int_{\Omega} \mathbf{B}^{\mathbf{T}}\mathbf{D}\mathbf{B}\, d\Omega - \int_{\Gamma_d} \mathbf{N}^T(\mathbf{n}^T\mathbf{D}\mathbf{B})\, d\Gamma - \int_{\Gamma_d} (\mathbf{n}^T\mathbf{D}\mathbf{B})^T\mathbf{N}\, d\Gamma + \beta \int_{\Gamma_d} \mathbf{N}^T\mathbf{N}\, d\Gamma\right)\mathbf{u} &= \\ \int_{\Omega} \mathbf{N}^T\mathbf{b}\, d\Omega - \int_{\Gamma_n} \mathbf{N}^T\mathbf{t}\, d\Gamma - \int_{\Gamma_d} (\mathbf{n}^T\mathbf{D}\mathbf{B})^T\mathbf{u}_d\, d\Gamma + \beta \int_{\Gamma_d} \mathbf{N}^T\mathbf{u}_d\, d\Gamma & \end{aligned} \tag{7}$$

4 NUMERICAL EXAMPLE

Figure 1 depicts a cantilever of length 8 ul, height 2 ul and unit thickness, subjected to a parabolic load totalizing 2 uf acting on its right edge. On it's left edge displacements were imposed according to the analytical solution of this problem. The elastic material has $E = 1000$ uf/ua and $\nu = 0,25$. Four discrete models were considered, namely, $5 \times 17, 9 \times 33, 17 \times 65, 33 \times 128$, where $ny \times nx$ are the number of nodes in the $y-$ and $x-$ directions, respectively. Also, 4 values of β were adopted, i.e., $\beta = \{10^3, 10^4, 10^5, 10^6\}$. Plane stress conditions are assumed. The EFG employed a cubic spline as weight function, and linear approximation with dilation parameter $\rho_x = \rho_y = 1,05h$, where h is the distance between nodes.

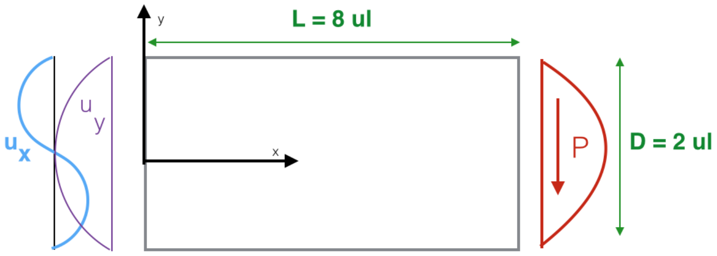

Figure 1: Cantilever: in blue the $u_x(0;y)$ pattern and in purple the $u_y(0;y)$ pattern

In Fig. 2 it is shown the relative error, using the $\mathcal{L}^2$-norm, of the displacements on the essential boundary for the 3 methods against the distance between nodes h. It is quite noticiable that the error in the penalty method does not diminish as the discretization is refined for a fixed β, whereas the other two methods shown an improvement.

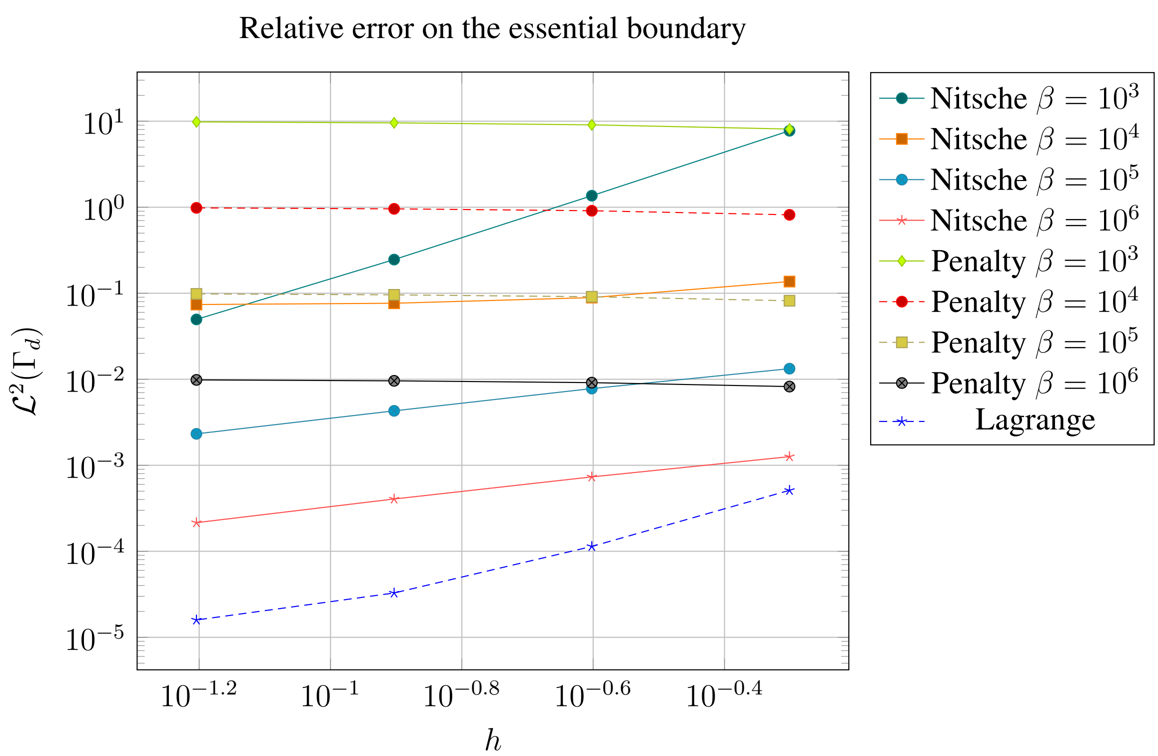

Figure 2: Nitsche vs Penalty vs Lagrange: Relative error on the essential boundary

Figure 3 shows the horizontal displacement at the left edge for the three methods of the model 9×33. As it can be seen, for a small value of β (10^3), the penalty method is unable to reproduce the essential boundary condition for the horizontal displacement, whereas for the Lagrange and Nitsche methods the performace was satisfactory.

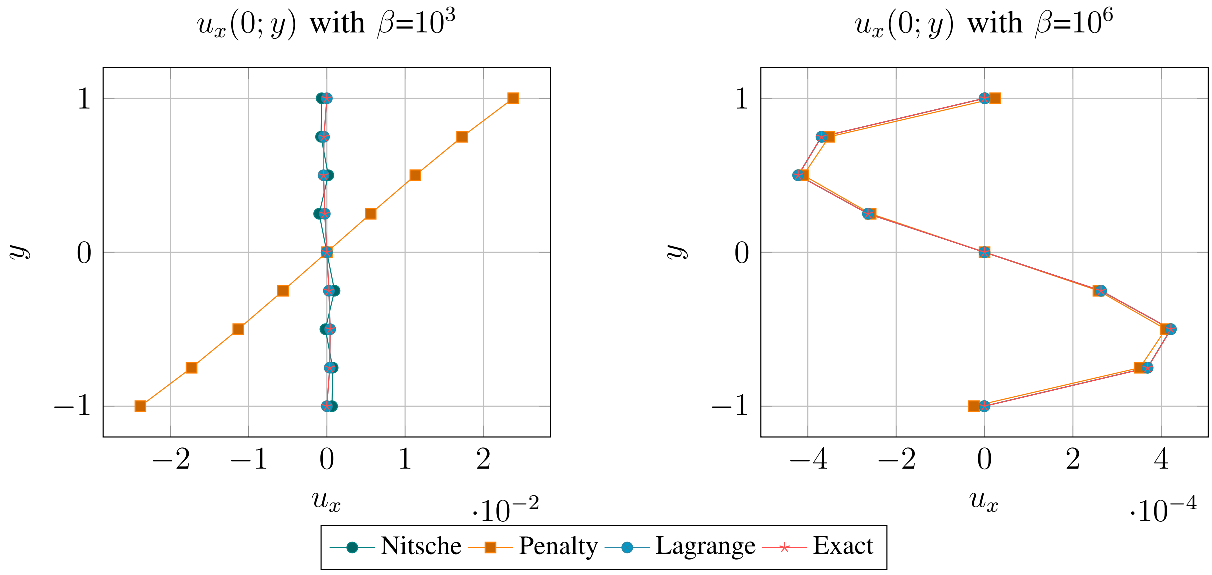

Figure 3: Model 9×33 $u_x(0;y)$ **with** $\beta = 10^3$ **(left) and** $\beta = 10^6$ **(right)**

5 CONCLUSION

The results of this work show that the Nitsche's method is an excelent option to imposing essential boundary conditions, is variationally consistent and gives similar results when compared to the Lagrange Multipliers method, however using a smaller scalar parameter. One disadvantage, though, is that its formulation is not straightforward and problem dependent. As expected, for a given accuracy, the penalty method demands a larger value for the scalar parameter, which sometimes may ruin the conditioning of the system of equations. However, this is still a work in progress, and an implementation considering a physically nonlinear analysis is being carried on.

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