Mathematical Model Of a Collision Based On a Spring-Mass-Damper System With a Nonlinear Spring Behavior

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Abstract. *This paper presents a mathematical model of a collision were the phenomenon will be simplified by a spring-mass-damper model. The spring will be considered to have an elasto-plastic behavior, which states that the spring suffers a permanent deformation after a force application. The responses of the model will be obtained analytically and by numerical approximation. The model proposed in this paper allows one to obtain parameters of the system, and then compare to a full-scale experiment to test its suitability with it. The nonlinear behavior of the system will be characterized by analyzing the phase space diagram.*

Keywords: *Mechanical vibrations, Collision model, Numerical simulation, Mathematical model*

1 INTRODUCTION

In engineering, a great number of phenomena are modeled by linear models, which in many cases provide reasonable results and with good agreement with the experimental data. There are, however, some phenomena that cannot be described by linear systems. Therefore, the use of nonlinear methods to model these kind of phenomena are needed in order to produce more reliable results (Strogatz S., 1994; Tél T. et al., 2006).

In this paper, a mathematical model of a collision is presented, where the phenomenon is simplified by a spring-mass-damper model. We will analyze the phenomenon internally and make a model with the purpose of fitting its responses with an experimental collision signal properly measured. Our interest is at the very short time which the two bodies collide and separate. The model has applications in cases where the behavior of a colliding body must be known and it's difficult to make an experimental process for such purpose. Pawlus (2013) shows a mathematical modeling of a vehicle crash and its importance in the collision analysis.

Since is about a very complex phenomenon, the treatment with only linear parameters is ineffective. Thus the elasto-plastic behavior, which is a nonlinear force-deflection characteristic, will be considered for the spring of the system. As will be shown, this nonlinearity is added by just changing the arrangement of a regular spring-mass-damper system. The model's responses will be obtained analytically and by numerical approximation, for this last case the Runge-Kutta algorithm will be used.

2 Background

2.1 Spring-Mass-Damper System

To analyze the collision, the phenomenon will be simplified as a spring-mass-damper system, where the damper is added after a certain deformation of the spring. Figure 1 shows the spring-mass-damper system proposed, where: c is the damper coefficient, k_U is the unloading spring stiffness, k_L is the loading spring stiffness, v_0 is the initial velocity, m is the mass of the system, x_m is the maximum deformation.

The nonlinearity is added by considering two spring stiffness for the system: a loading and an unloading. The relationship between these two parameters as well as the elasto-plastic behavior is described in section 2.3. In addiction, the damper is added with the purpose of letting the model more real, since there is no movement without energy dissipation.

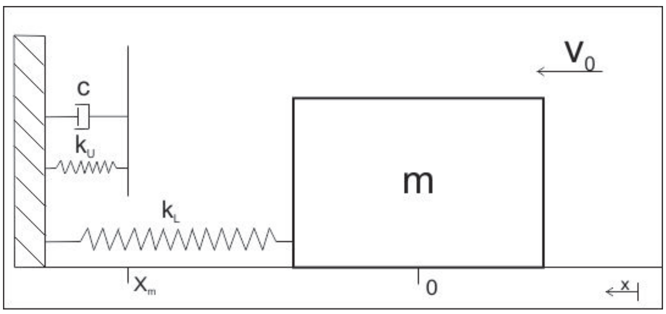


Figure 1: Spring-Mass-Damper Model

2.2 Model's Establishment

In order to establish the mathematical model, the analysis will be divided by the two kinds of movement present in the system: an undamped oscillatory one which goes until the velocity becomes zero, and an underdamped one which starts right after and goes until the bodies separate. movement will be a non-decayed oscillatory one, and the second part will be an underdamped system with an exponential decay.

Undamped System

The first part of the movement includes the time when the bodies collide and when their relative velocity becomes zero. The use of only a spring represents well this part, as well as facilitates the analysis making less complicated to obtain the parameters. The equation of movement of the first part can be written as,

$$m\ddot{x}(t) + kx(t) = 0 \tag{1}$$

where $x(t)$ is the displacement of the mass in function of time, m is the system's mass, k is the spring stiffness and the dots represent derivative with respect of time.

Since the analysis starts when the two bodies collide, the initial conditions for the first part of the movements are: $x(0) = 0$ and $\dot{x}(0) = v_0$. Thus the solution of Eq. (1) is (Rao, 2009),

$$x(t) = \frac{v_0}{\omega_n}\sin(\omega_n t) \tag{2}$$

where ω_n is the natural frequency at the loading part and is defined as,

$$\omega_n = \sqrt{\frac{k}{m}}. \tag{3}$$

The velocity and acceleration of the mass can be obtained by differentiating Eq. (1) with respect to time, which gives,

$$\dot{x}(t) = v_0\cos(\omega_n t) \tag{4}$$

$$\ddot{x}(t) = -\omega_n v_0 \sin(\omega_n t). \tag{5}$$

Equation (4) and (5) represent the velocity and the acceleration of the mass, respectively. The time when the velocity becomes zero for an undamped system is defined as (Huang, 2002),

$$t_m = \frac{\pi}{2\omega_n}. \tag{6}$$

The spring stiffness of the system can be obtained by,

$$k = \left(\frac{v_0}{x_m}\right)^2 m. \tag{7}$$

As one can see by Eq. (7), the spring stiffness can be obtained by knowing the initial velocity, the maximum deformation and the system's mass. These quantities can be easily measured. Moreover, by knowing the spring stiffness the time when the velocity becomes zero (t_m) can be obtained by Eq. (6).

Underdamped System

The second part of the movement starts after the velocity becomes zero and goes until the bodies separate. This part is a damped oscillatory movement. The underdamped case will be considered as this case represents well a great number of oscillation phenomena in engineering. The equation of movement for this part is,

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = 0 \tag{8}$$

which now have a damper coefficient c. Equation (8) can be written as (Bachalandran, 2009),

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = 0 \tag{9}$$

where ζ is the damping factor, which in the underdamped case have the value between $0<\zeta<1$. The initial conditions for the second part of the movement are: $x(0) = x_0$ and $\dot{x}(0) = 0$. The solution of Eq. 9 is (Rao, 2009),

$$x(t) = x_p + x_0 e^{-\zeta\omega_n t}\left[\cos(\omega_n\sqrt{1-\zeta^2}t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_n\sqrt{1-\zeta^2}t)\right]. \tag{10}$$

Please note that the initial displacement (x_0) in these responses is the difference between the maximum deformation (x_m) and the permanent deformation (x_p). As one can see on the model's illustration (Fig. 1), after the system reach the maximum deformation it will oscillate around the permanent deformation, that's why the term x_p was added on the right of Eq. (10). The velocity and the acceleration can be obtained by differentiating Eq. (10) with respect of time, which gives,

$$\dot{x}(t) = -\frac{x_0\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}t) \tag{11}$$

$$\ddot{x}(t) = x_0\omega_n^2 e^{-\zeta\omega_n t}\left[\frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_n\sqrt{1-\zeta^2}t) - \cos(\omega_n\sqrt{1-\zeta^2}t)\right]. \tag{12}$$

Since there is no theoretical way to obtain the value of the damping factor (ζ), its value must be measured or adopted *a priori*, which will depend on the critical analysis of the phenomenon, especially the analysis of the material of the bodies.

2.3 Elasto-Plastic Spring

All the springs exhibit an elasto-plastic characteristic, which means that after the application and the release of an excessive force there will be a change in the spring stiffness, having then a loss of the elastic energy due to the plastic deformation (Pawlus et al., 2010).

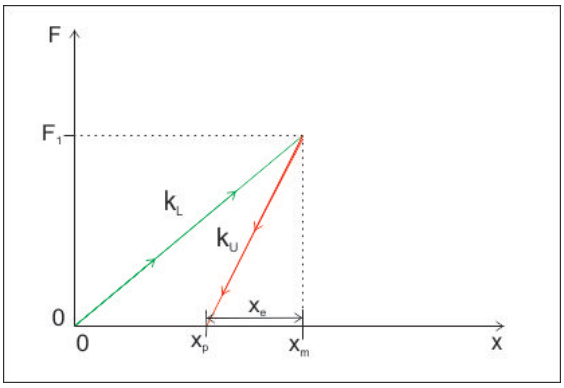


Figure 2: Nonlinear Force-Deflection Characteristic

In the elasto-plastic analysis, we let the spring with two spring stiffness values: a loading spring stiffness (k_L), which is the stiffness during the collision. and an unloading spring stiffness (k_U), being $k_L{<}k_U$. Figure 2 shows the elasto-plastic behavior.

As one can see in Fig. 2, the spring have got a permanent deformation, given by x_p, and a elastic recovery, given by x_e, which represents the elastic energy recovered by the spring. The relation between the loading and the unloading spring stiffness is written as the following,

$$k_U = k_L \left(\frac{x_m}{x_e}\right). \tag{13}$$

Equation (13) will be used to find the unloading spring stiffness.

3 Simulations

The initial parameters that will be used for the simulations are presented in Table 1. For the numerical approach the Runge-Kutta algorithm was used. All the plots were made by using the software Matlab.

3.1 Analytical Solution

In the analytical solution, the parameters will be obtained and the responses (displacement, velocity and acceleration) will be evaluated by the analytical solution of the equations of motion. First, we obtain the loading spring stiffness by using Eq. (7), which gives

$$k_L = \left(\frac{(0.3889\, m/s)}{(0.01\, m)}\right)^2 (1.0\, kg) = 1512.4\, N/m$$

thus the loading natural frequency and the time when the velocity becomes zero become, respectively,

$$\omega_{nL} = \sqrt{\frac{(1512.4\, N/m)}{(1.0\, kg)}} = 38.89\, rad/s$$

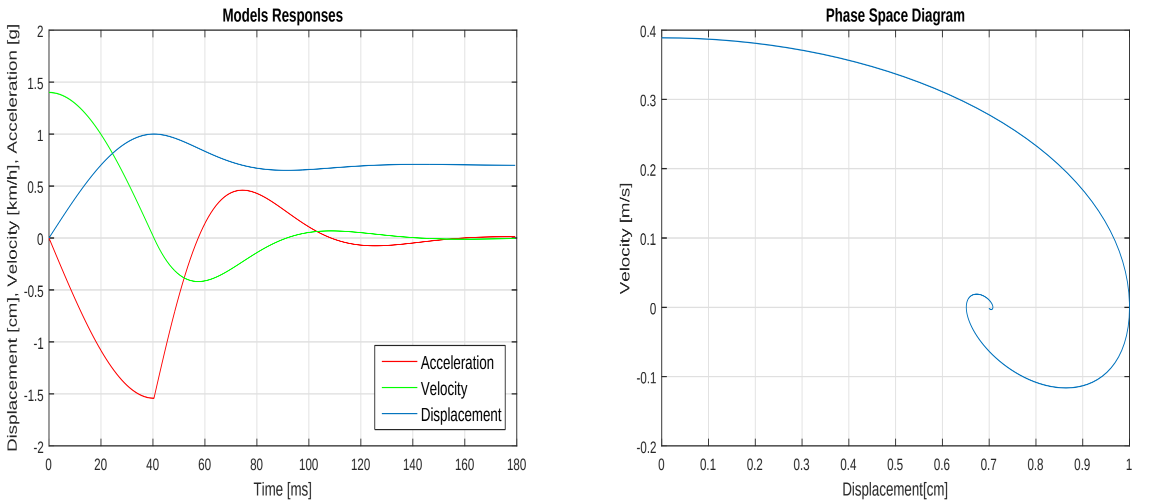
$$t_m = \frac{\pi}{2 \times 38.89\, rad/s} = 0.0404\, s.$$

Therefore, the system will behave as an undamped system until $0.0404\, s$, and after that as an underdamped system. The unloading spring stiffness and natural frequency can be obtained by Eqs. (13) and (3), respectively,

$$k_U = 1512.4\, N/m \times \left(\frac{0.01\, m}{0.003\, m}\right) = 5041.3\, N/m$$

Table 1: Initial Parameters

Parameter	Value
Initial Velocity (v_0)[m/s]	0.3889
Mass of the System (m)[kg]	1.0000
Maximum Deformation (x_m)[m]	0.0100
Permanent Deformation (x_d)[m]	0.0070



(a) (b)

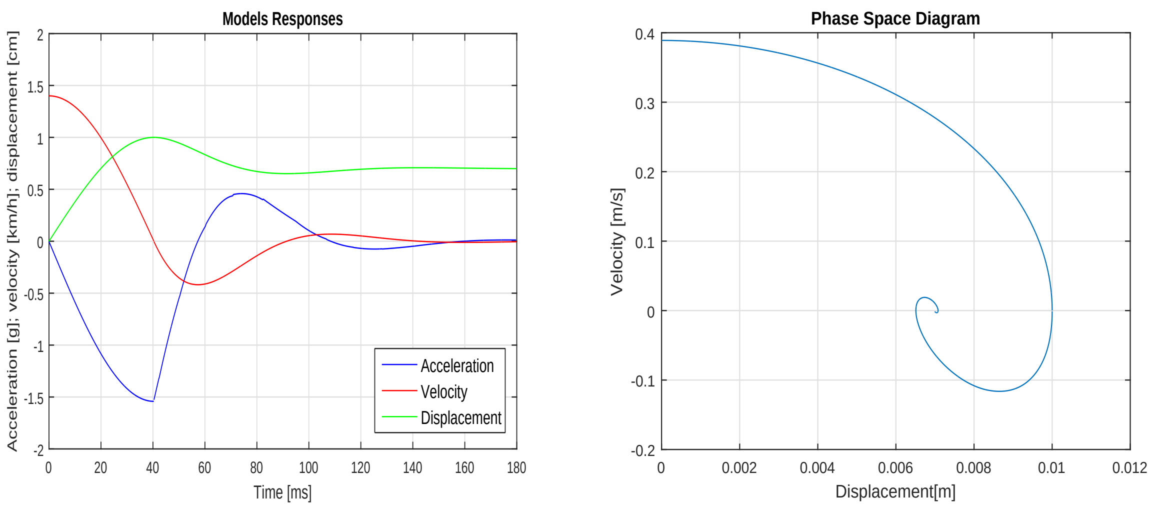
Figure 3: Model's Responses, Figure 3a, and Phase Space Diagram, Figure 3b (Analytical Solution)

$$\omega_{nU} = \sqrt{\frac{(5041.3\ N/m)}{(1.0\ kg)}} = 71.0\,rad/s.$$

The value of the damping factor will be considered as $\zeta = 0.5$ for the simulation. The responses can now be plotted and they are shown in Fig. 3.

Our collision analysis is till the velocity reaches its maximum value, which occur at $57.4ms$, because we only consider one loading and unloading for the spring, not a reloading. The rest of the simulation is what one would get if the system continue to behave as an underdamped one.

We can note by analyzing the phase space diagram (Figure 3b) that the system behaves as a nonlinear one. The distances between the trajectories are not constant and do not converge to



(a) (b)

Figure 4: Model's Responses, Figure 4a, and Phase Space Diagram, Figure 4b (Numerical Solution)

the center of the ellipse which is a characteristic of a nonlinear system.

3.2 Numerical Solution

For the numerical solution, the solver *ode45* of the software Matlab was used. This solver executes the forth order Runge-Kutta algorithm. The solver will be used to solve eqs. (1) and (9). Moreover, the algorithm will give the displacement of the mass, thus the velocity and the acceleration will be obtained by numerical differentiation.

Using the parameters obtained in the previous section, the responses can be evaluated and they are showed in Fig. 4. As one can see, the numerical responses were very close to the analytical ones, proving the effectiveness of the numerical approach.

4 Conclusions

We presented a mathematical model of a collision with the purpose to fit its responses with an experimental signal. Only numerical simulations were presented. In order to validate the model an experimental process must be done. The model has practical as well as educational applications. For example, it can be introduced in undergraduate courses as a case study. In addiction, an extensive work has been spend to develop a vehicle crash mathematical models. References (Pawlus W., 2010) and (Klausen A., 2014) treat this issue.

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