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MODELING A ROCKET ELASTIC STRUCTURE AS A BECK'S COLUMN UNDER FOLLOWER FORCE

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Abstract. *It is intended, in this paper, to develop a mathematical model of an elastic space rocket structure as a Beck's column excited by a follower (or circulatory) force. This force represents the rocket motor thrust that should be always in the direction of the tangent to the structure deformed axis at the base of the vehicle.*

We present a simplified two degree of freedom rigid bars discrete model. Its system of two second order nonlinear ordinary differential equations of motion are derived via Lagrange's energy method, allowing for a general understanding of the main characteristics of the problem. The proposed equations consider up to third order (cubic) inertia, stiffness and forcing terms.

Among other rich nonlinear dynamic behavior of this model, depending on parameters and initial conditions choices, either stable or unstable limit cycle solutions are possible. The unstable solution is, of course, an interesting simple example of flutter instability.

Keywords: *Beck's column, follower force, nonlinear dynamics, flutter.*

1 INTRODUCTION

Launcher vehicles (popularly known as rocket) are essential devices to carry loads from the surface of the Earth to some orbit around it, whatever is the current space mission. The vehicle, like any other physical body, is not absolutely rigid, so that the structural behavior, via excitation by external loads, tends to affect the flight dynamics.

It is intended, in this paper, to develop a mathematical model of an elastic space rocket structure as a Beck's column excited by a follower (or circulatory) force. This force represents the rocket motor thrust that should be always in the direction of the tangent to the structure deformed axis at the base of the vehicle.

We present a simplified two degree of freedom rigid bars discrete model. Its system of two second order nonlinear ordinary differential equations of motion are derived via Lagrange's energy method, allowing for a general understanding of the main characteristics of the problem. The proposed equations consider up to third order (cubic) inertia, stiffness and forcing terms.

Among other rich nonlinear dynamic behavior of this model, depending on parameters and initial conditions choices, either stable or unstable limit cycle solutions are possible. The unstable solution is, of course, an interesting simple example of flutter instability.

Our model compares very well with previous work by Mazzilli (1988) in the context of Civil Engineering. Timoshenko (2009) also presents an analytical solution of the fourth order partial differential equation of motion of Beck's column.

2 PHYSICAL MODEL

2.1 The physical model

Figure 1 is our simplified physical model of the structure of a launcher vehicle. It is constructed of two rigid massless bars $\overline{AB}$ and $\overline{BC}$, L_1 and L_2 long, respectively, pinned to nodes A and B. Displacements are restricted at point A. We consider lumped masses M_1, M_2 and M_3 attached to nodes A and B where torsional springs k_1 e k_2 provide elastic restoring forces.

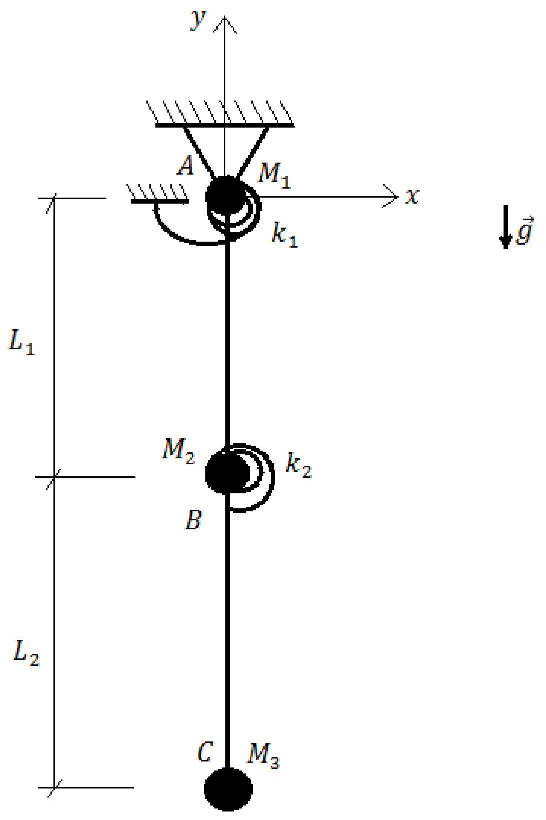

Figure 1. Physical model at rest position.

2.2 Modeling hypothesis

1. It is adopted $L_1 = L_2 = L$.
2. The bars are rigid and massless.
3. Lumped masses M_1, M_2 and M_3 represent the actual masses of half the bars connected to that point. If $2m$ is the mass of each bar, $M_1 = M_3 = m$ and $M_2 = 2m$.
4. We consider the stiffness of the torsional springs to represent the elastic properties of the continuous structure. It is adopted $k_1 = k_2 = k$.
5. The adopted inertial reference is point A, origin of an orthonormal basis $\mathcal{B}_g \equiv (\hat{i}, \hat{j})$, where $\hat{i} \equiv \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$ and $\hat{j} \equiv \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$ are the unit vectors.
6. Motions are restricted to the xy plan.
7. Initially, only self-weight forces act. This is the fundamental static equilibrium configuration of the system, representing the vehicle at rest in its launch platform.

2.3 Excitation

Let $\vec{F}$ be a follower (circulatory) non conservative force applied to C, in the direction of bar $\overline{BC}$. This force models the rocket's thrust force due to combustion gases expansion at the motors in the basis of the vehicle. We do not consider, in this model, its dependence on time.

The action of force $\vec{F}$ applied to C excites the system to depart from its fundamental equilibrium position. The problem is now similar to an excited inverted double-pendulum with elastic properties.

The equations of motion could be derived via Newton's second law vector approach, but this method is quite cumbersome in this case. Thus a Lagrangian scalar energy scheme is preferable. Our generalized coordinates are angular displacements θ_1 e θ_2 of bars $\overline{\text{AB}}$ e $\overline{\text{BC}}$, computed from their original vertical equilibrium positions. They are, of course, implicitly time dependent, that is, $\theta_1 = \theta_1(t)$ and $\theta_2 = \theta_2(t)$. We denote $\theta_1(t) \equiv q_1(t)$ and $\theta_2(t) \equiv q_2(t)$. Nonzero values represent the vehicle in flight conditions as represented in Fig, 2.

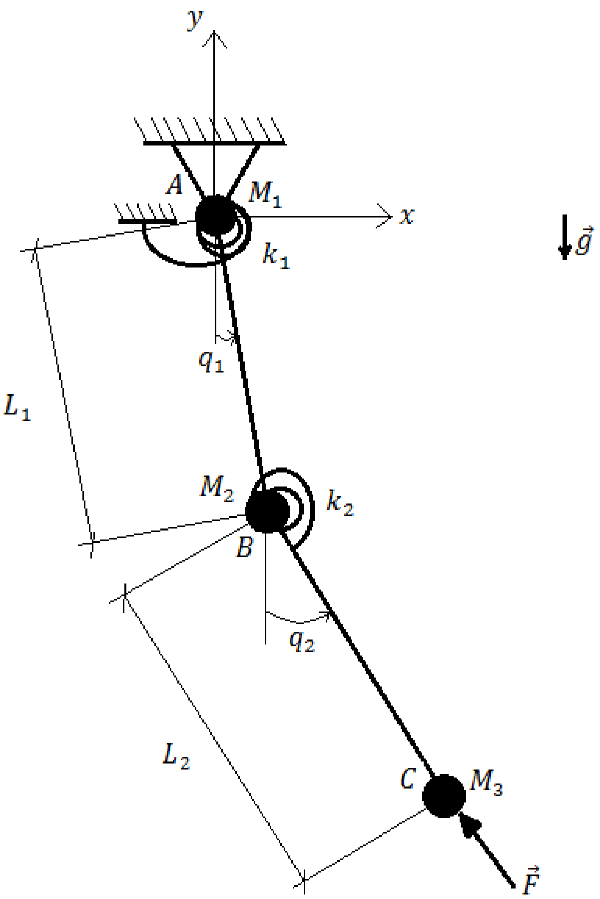

Figure 2. System motion under follower force $\vec{\mathrm{F}}$.

3 MATHEMATICAL MODEL

3.1 Kinematics

3.1.1 Position vectors of the lumped masses

Time dependent vector position of the masses along their motions are given by Eqs. (1), (2) and Eq. (3):

$$\vec{r}_1 = \vec{0} \tag{1}$$

$$\vec{r}_2 = L\left(\sin q_1 \hat{i} - \cos q_1 \hat{j}\right) \tag{2}$$

$$\vec{r}_3 = L\left[\left(\sin q_1 + \sin q_2\right)\hat{i} - \left(\cos q_1 + \cos q_2\right)\hat{j}\right] \tag{3}$$

3.1.2 Velocity vectors of the lumped masses

Time derivatives, denoted by a superposed dot, lead to velocity vectors given by Eqs. (4), (5) and Eq. (6):

$$\dot{\vec{r}}_1 = \vec{0} \tag{4}$$

$$\dot{\vec{r}}_2 = L\left(\dot{q}_1 \cos q_1 \hat{i} + \dot{q}_1 \sin q_1 \hat{j}\right) \tag{5}$$

$$\dot{\vec{r}}_3 = L\left[\left(\dot{q}_1 \cos q_1 + \dot{q}_2 \cos q_2\right)\hat{i} + \left(\dot{q}_1 \sin q_1 + \dot{q}_2 \sin q_2\right)\hat{j}\right] \tag{6}$$

3.2 Approximations

We adopt a third order truncated polynomial approximation to the sinusoidal functions, given by Maclaurin series around zero equilibrium conditions, considering not very large displacement angles.

$$\begin{cases} \cos q_i \cong 1 - \dfrac{q_i^{\,2}}{2} \\ \sin q_i \cong q_i - \dfrac{q_i^{\,3}}{6} \end{cases} \tag{7}$$

3.3 Energy computation

In the following equations we neglect terms of higher order than third.

3.3.1 Kinetic energy

Kinetic energy of this system is given by Eq. (8):

$$T \cong \frac{1}{2} m L^2 \left[3\dot{q}_1^{\,2} + \dot{q}_2^{\,2} + 2\left(\dot{q}_1 \dot{q}_2 - \dot{q}_1 \dot{q}_2 \frac{q_1^{\,2}}{2} - \dot{q}_1 \dot{q}_2 \frac{q_2^{\,2}}{2} + \dot{q}_1 \dot{q}_2 q_1 q_2 \right)\right] \tag{8}$$

3.3.2 Total potential energy

The total potential energy is

$$V = U - W \tag{9}$$

where U is the strain energy of the springs and W is the work of the conservative forces acting on the system, namely the self-weight of the lumped masses.

The strain energy, in terms of the generalized coordinates is given by Eq. (10):

$$U = \frac{1}{2}k\left(2q_1^{\ 2} + q_2^{\ 2} - 2q_1\ q_2\right) \tag{10}$$

The work of the conservative forces in terms of the generalized coordinates is given by Eq. (11):

$$W = \frac{1}{2}mgL\left(3q_1^{\ 2} + q_2^{\ 2}\right) \tag{11}$$

Thus, the total potential energy is

$$V = \frac{1}{2}\left[k\left(2q_1^{\ 2} + q_2^{\ 2} - 2q_1\ q_2\right) - mgL\left(3q_1^{\ 2} + q_2^{\ 2}\right)\right] \tag{12}$$

3.4 Derivation of the equations of motion

Next, we apply Euler-Lagrange equations, Eqs. (13) and (14), following Brasil (1996):

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial}{\partial \dot{q}_i}\mathcal{L}\right) - \frac{\partial}{\partial q_i}\mathcal{L} = F_i^{nc} \tag{13}$$

where

$$F_i^{nc} = \sum_{j=1}^{N} \vec{f}_j^{\ nc} \bullet \frac{\partial}{\partial q_i}\vec{r}_j \tag{14}$$

i = 1,2 refer to the generalized coordinates and j = 1,2 e 3 to the nodes where lumped masses are connected.

The Lagrangian, a functional of the generalized coordinates, is given by Eq. (15):

$$\mathcal{L} = \mathcal{L}\left(q_1, q_2, \dot{q}_1, \dot{q}_2\right) = T\left(q_1, q_2, \dot{q}_1, \dot{q}_2\right) - V\left(q_1, q_2\right) \tag{15}$$

The two differential equations of motion are of the form:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{dt}}\left[\dfrac{\partial}{\partial \dot{q}_1}\mathcal{L}(q_1, q_2, \dot{q}_1, \dot{q}_2)\right] - \dfrac{\partial}{\partial q_1}\mathcal{L}(q_1, q_2, \dot{q}_1, \dot{q}_2) = F_1^{nc} \\ \dfrac{\mathrm{d}}{\mathrm{dt}}\left[\dfrac{\partial}{\partial \dot{q}_2}\mathcal{L}(q_1, q_2, \dot{q}_1, \dot{q}_2)\right] - \dfrac{\partial}{\partial q_2}\mathcal{L}(q_1, q_2, \dot{q}_1, \dot{q}_2) = F_2^{nc} \end{cases} \tag{16}$$

The Lagrangian functional in terms of energies is:

$$\mathcal{L}(q_1, q_2, \dot{q}_1, \dot{q}_2) = T(q_1, q_2, \dot{q}_1, \dot{q}_2) - U(q_1, q_2) + W(q_1, q_2) \tag{17}$$

rendering Eq. (18):

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{dt}}\left[\dfrac{\partial}{\partial \dot{q}_1}T(q_1, q_2, \dot{q}_1, \dot{q}_2)\right] - \dfrac{\partial}{\partial q_1}T(q_1, q_2, \dot{q}_1, \dot{q}_2) + \dfrac{\partial}{\partial q_1}U(q_1, q_2) - \dfrac{\partial}{\partial q_1}W(q_1, q_2) = F_1^{nc} \\ \dfrac{\mathrm{d}}{\mathrm{dt}}\left[\dfrac{\partial}{\partial \dot{q}_2}T(q_1, q_2, \dot{q}_1, \dot{q}_2)\right] - \dfrac{\partial}{\partial q_2}T(q_1, q_2, \dot{q}_1, \dot{q}_2) + \dfrac{\partial}{\partial q_2}U(q_1, q_2) - \dfrac{\partial}{\partial q_2}W(q_1, q_2) = F_2^{nc} \end{cases} \tag{18}$$

3.4.1 Derivation

For the first generalized coordinate:

$$\frac{\mathrm{d}}{\mathrm{dt}}\left[\frac{\partial}{\partial \dot{q}_1}T(q_1, q_2, \dot{q}_1, \dot{q}_2)\right] = mL^2\left[3\ddot{q}_1 + \ddot{q}_2 - \frac{\ddot{q}_2}{2}(q_1 - q_2)^2 + \dot{q}_2{}^2(q_1 - q_2) - \dot{q}_1\dot{q}_2(q_1 - q_2)\right] \tag{19}$$

$$-\frac{\partial}{\partial q_1}T(q_1, q_2, \dot{q}_1, \dot{q}_2) = mL^2\dot{q}_1\dot{q}_2(q_1 - q_2) \tag{20}$$

$$\frac{\partial}{\partial q_1}U(q_1, q_2) = k(2q_1 - q_2) \tag{21}$$

$$-\frac{\partial}{\partial q_1}W(q_1, q_2) = -3mgLq_1 \tag{22}$$

For the second generalized coordinate:

$$\frac{\mathrm{d}}{\mathrm{dt}}\left[\frac{\partial}{\partial \dot{q}_2}T(q_1, q_2, \dot{q}_1, \dot{q}_2)\right] = mL^2\left[\ddot{q}_1 + \ddot{q}_2 - \frac{\ddot{q}_1}{2}(q_1 - q_2)^2 - \dot{q}_1{}^2(q_1 - q_2) + \dot{q}_1\dot{q}_2(q_1 - q_2)\right] \tag{23}$$

$$-\frac{\partial}{\partial q_2}T(q_1, q_2, \dot{q}_1, \dot{q}_2) = -mL^2\dot{q}_1\dot{q}_2(q_1 - q_2) \tag{24}$$

$$\frac{\partial}{\partial q_2}U(q_1, q_2) = -k(q_1 - q_2) \tag{25}$$

$$-\frac{\partial}{\partial q_2}W(q_1, q_2) = -mgLq_2 \tag{26}$$

3.4.2 Generalized non-conservative forces

The generalized non conservative forces, due to the follower force $\vec{F}$ are shown in Eqs. (27) and (28):

$$\vec{f}_1^{nc} = \vec{f}_2^{nc} = \vec{0} \tag{27}$$

$$\vec{f}_3^{nc} = F\left(-\sin q_2 \hat{i} + \cos q_2 \hat{j}\right) \tag{28}$$

Plugging Eqs. (3), (27) and (28) in Eq. (14), and considering approximations given by Eq. (7):

$$F_1^{nc} \cong -FL\left(-q_1 + q_2 + \frac{q_1^{\,3}}{6} - \frac{q_2^{\,3}}{6} + \frac{q_1 q_2^{\,2}}{2} - \frac{q_2 q_1^{\,2}}{2}\right) \tag{29}$$

$$F_2^{nc} = 0 \tag{30}$$

where F is the scalar value of the follower force.

3.5 Matrix equations of motion

The equations of motion of this system may be presented in matrix form as

$$\begin{aligned}&[M]\{\ddot{q}_i(t)\} + [C]\{\dot{q}_i(t)\} + [K]\{q_i(t)\} = \{F_i^{nc}(t)\}\\ &i = 1,2\end{aligned} \tag{31}$$

Where $[M]$ is the inertia or mass matrix, $[C]$ is the equivalent damping matrix and $[K]$ is the stiffness matrix. $\{q_i(t)\}$, $\{\dot{q}_i(t)\}$ and $\{\ddot{q}_i(t)\}$ are, respectively the generalized position, velocity and acceleration vectors. $[M]\{\ddot{q}_i(t)\}$ are inertia forces, $[C]\{\dot{q}_i(t)\}$ dissipative forces, $[K]\{q_i(t)\}$ restoring forces and $\{F_i^{nc}(t)\}$ is the vector of non-conservative generalized forces.

From Eqs. (19) at (26), we get:

$$[M] = mL^2\begin{bmatrix} 3 & 1-\frac{1}{2}(q_1 - q_2)^2 \\ 1-\frac{1}{2}(q_1 - q_2)^2 & 1 \end{bmatrix} \tag{32}$$

the symmetric inertia matrix that may be put in form:

$$[M] = [\tilde{M}] + [M_q] \tag{33}$$

where $\left[\tilde{M}\right]$ is the constant part of $\left[M\right]$, and $\left\lfloor M_q \right\rfloor = \left\lfloor M_q(q_1, q_2) \right\rfloor$ is a part of the matrix that depends on the time varying generalized coordinates q_1 and q_2 .Thus:

$$\left[\tilde{M}\right] = mL^2 \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \tag{34}$$

$$\left[M_q\right] = \left[M_q(q_1, q_2)\right] = mL^2 \begin{bmatrix} 0 & -\frac{1}{2}(q_1 - q_2)^2 \\ -\frac{1}{2}(q_1 - q_2)^2 & 0 \end{bmatrix} \tag{35}$$

The symmetric equivalent damping matrix:

$$\left[C\right] = \left[C(q_1, q_2, \dot{q}_1, \dot{q}_2)\right] = mL^2 \begin{bmatrix} 0 & \dot{q}_2(q_1 - q_2) \\ -\dot{q}_1(q_1 - q_2) & 0 \end{bmatrix} \tag{36}$$

We note that $\left[C\right] = \left[C(q_1, q_2, \dot{q}_1, \dot{q}_2)\right]$ depends on the generalized coordinates q_1 and q_2, and their time derivatives.

Finally, the symmetric stiffness matrix is:

$$\left[K\right] = \begin{bmatrix} 2k - 3mgL & -k \\ -k & k - mgL \end{bmatrix} \tag{37}$$

Considering $\left[K_U\right]$ the elastic component (related to the strain energy of the torsional springs) and $\left[K_W\right]$ a component related to the conservative forces actin on the system, we have:

$$\left[K\right] = \left[K_U\right] + \left[K_W\right] \tag{38}$$

Thus:

$$\left[K_U\right] = k \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \tag{39}$$

$$\left[K_W\right] = -mgL \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \tag{40}$$

Both $\left[K_U\right]$ and $\left[K_W\right]$ are also symmetric.

The final matrix equations of motion are:

$$mL^2\begin{bmatrix} 3 & 1-\frac{1}{2}(q_1-q_2)^2 \\ 1-\frac{1}{2}(q_1-q_2)^2 & 1 \end{bmatrix}\begin{Bmatrix}\ddot{q}_1(t)\\ \ddot{q}_2(t)\end{Bmatrix}+$$

$$mL^2\begin{bmatrix} 0 & \dot{q}_2(q_1-q_2) \\ -\dot{q}_1(q_1-q_2) & 0 \end{bmatrix}\begin{Bmatrix}\dot{q}_1(t)\\ \dot{q}_2(t)\end{Bmatrix}+\begin{bmatrix} 2k-3mgL & -k \\ -k & k-mgL \end{bmatrix}\begin{Bmatrix}q_1(t)\\ q_2(t)\end{Bmatrix}= \tag{41}$$

$$-FL\begin{Bmatrix} -q_1+q_2+\frac{q_1^{\ 3}}{6}-\frac{q_2^{\ 3}}{6}+\frac{q_1q_2^{\ 2}}{2}-\frac{q_2q_1^{\ 2}}{2} \\ 0 \end{Bmatrix}$$

or:

$$\left[\left[\tilde{M}\right]+\left[M_q(q_1,q_2)\right]\right]\{\ddot{q}_i(t)\}+\left[C(q_1,q_2,\dot{q}_1,\dot{q}_2)\right]\{\dot{q}_i(t)\}+\left[\left[K_U\right]+\left[K_W\right]\right]\{q_i(t)\}=\{F_i^{nc}(t)\} \tag{42}$$

$i=1{,}2$

and:

$$mL^2\left[\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}+\begin{bmatrix} 0 & -\frac{1}{2}(q_1-q_2)^2 \\ -\frac{1}{2}(q_1-q_2)^2 & 0 \end{bmatrix}\right]\begin{Bmatrix}\ddot{q}_1(t)\\ \ddot{q}_2(t)\end{Bmatrix}+$$

$$mL^2\begin{bmatrix} 0 & \dot{q}_2(q_1-q_2) \\ -\dot{q}_1(q_1-q_2) & 0 \end{bmatrix}\begin{Bmatrix}\dot{q}_1(t)\\ \dot{q}_2(t)\end{Bmatrix}+\left[k\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}-mgL\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\right]\begin{Bmatrix}q_1(t)\\ q_2(t)\end{Bmatrix}= \tag{43}$$

$$-FL\begin{Bmatrix} -q_1+q_2+\frac{q_1^{\ 3}}{6}-\frac{q_2^{\ 3}}{6}+\frac{q_1q_2^{\ 2}}{2}-\frac{q_2q_1^{\ 2}}{2} \\ 0 \end{Bmatrix}$$

4 CONCLUSIONS AND FUTURE WORK

We presented the derivation of a two degrees of freedom lumped parameters mathematical model of an elastic rocket launching vehicle excited by a follower force due to the its motors thrust. Its system of two second order nonlinear ordinary differential equations of motion are derived via Lagrange’s energy method, allowing for a general understanding of the main characteristics of the problem. The proposed equations consider up to third order (cubic) inertia, stiffness and forcing terms.

In further work, numerical step-by-step time integration of the equations of motion of the mathematical model is carried out via Runge-Kutta fourth order algorithm in order to study the dynamic stability regions of the problem. For application of the Runge-Kutta integration scheme, a state space system of four first order nonlinear ordinary differential equations will be derived from the original equations of motion.

Among other rich nonlinear dynamic behavior of this model, it is shown that, depending on parameters and initial conditions choices, either stable or unstable limit cycle solutions are possible. The unstable solution is, of course, an interesting simple example of flutter instability.

Some thought will also be given to possible control strategies of the resulting vibrations, a crucial problem when real rocket motions are considered.

We will also solve the exact analytical solution to the fourth order partial differential equation governing the motion of an elastic space rocket structure under follower force excitation. Last, a Finite Element Method numerical model of the problem will be developed and solved in order to find the solution approached by the simple two degree of freedom model presented in this paper. Comparisons will be carried out and expected to be very good.

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