REFINING COMPLEX FREQUENCIES OF VIBRATING STRUCTURES

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Abstract. *In structural analysis, and related, the computation of natural frequencies of vibrating systems given by its mass, damping and stiffness matrices, when at least one of this matrices is dense and the matrix size is not prohibitively large, is usually done through reduction to a generalized first order system. The QZ iteration method is then applied to achieve the Generalized Schur form, from where all the natural frequencies, and associated mode shape vectors, can be taken from, as eigenvalues and eigenvectors of some mathematical problem. As a drawback, this reduction not only increases by a factor of 2 the sizes of the working matrices, but also allows important properties like positive definiteness (or semi-definiteness), and even symmetry, to be lost. Even further, it must be considered the fact that the condition number of the new problem (as an upper bound for the ratio between changes in the solution and changes in the data, by means of matrix and vector norms) usually increases, sometimes by a large amount. This work investigates how the final computation of complex-valued system eigenvalues (and then complex system frequencies) can take advantage of an iterative refinement technique, based in a shifted and inverted Krylov-type strategy that uses the original linear second-order matrices, in the same computational environment. The proposed strategy takes advantage that, in its original second-order form, the eigenvalue problem seams to be far better conditioned and well suited for high performance computations than in the generalized first-order form. The proposed strategy is applied to some test matrices from the Harwell-Boeing Collection of the Matrix Market website. Numerical examples using professional software are also provided.*

Keywords: *second-order, complex frequency, vibration, structure*

1 INTRODUCTION

Large, medium and even small mechanical vibrating structures can sometimes be regarded as second-order linear systems and therefore can be described, in the state-space form, by

$$M\ddot{q} + C\dot{q} + Kq = f, \tag{1}$$

for symmetric matrices $M, C, K \in \mathbb{R}^{n \times n}$. Those matrices are usually obtained from the Finite Element Method and are usually called, respectively *mass*, *damping* and *stiffness* matrices. This is the case of vibrating structures such as bridges, buildings, transmission towers, airplanes and others. M is usually positive definite, while K is usually positive semi-definite. For systems having damped vibrations, if C cannot be obtained, then it is usually assumed to fit in the *proportional damping* hypothesis:

$$C = \alpha M + \beta K \ , \ \alpha, \beta \in \mathbb{R}. \tag{2}$$

As a consequence, matrix C is also symmetric. This is a pleasant consequence for computational strategies, although sometimes it is not physically reasonable.

Mathematically, important properties regarding stability and control of system (1) can be determined by the nature of the solutions pairs (λ_i, x_i) of the *second order eigenvalue problem*

$$\lambda^2 M x + \lambda C x + K x = 0. \tag{3}$$

Computational strategies for problem (3) usually transform it into a generalized first order eigenvalue problem

$$\lambda B y = A y \tag{4}$$

by introducing new variables defined by $u = \lambda x, v = x, y = \begin{bmatrix} u \\ v \end{bmatrix}$ and matrices A and B in a few possible ways:

C1: to keep the symmetry of both matrices:

$$B = \begin{bmatrix} -M & 0 \\ 0 & K \end{bmatrix} \ , \ A = \begin{bmatrix} C & K \\ K & 0 \end{bmatrix} \tag{5}$$

C2: to keep B positive for M and K positive:

$$B = \begin{bmatrix} M & C \\ 0 & K \end{bmatrix} \ , \ A = \begin{bmatrix} 0 & -K \\ K & 0 \end{bmatrix} \tag{6}$$

C3: to keep B symmetric and positive for positive M and K:

$$B = \begin{bmatrix} M & 0 \\ 0 & K \end{bmatrix} \ , \ A = \begin{bmatrix} -C & -K \\ K & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \tag{7}$$

C4: to keep B positive for positive M (useful if K is singular):

$$B = \begin{bmatrix} M & C \\ 0 & M \end{bmatrix} \ , \ A = \begin{bmatrix} 0 & -K \\ M & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \tag{8}$$

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After the reduction to Eq. (4) is established, if B is positive definite, a very used approach that provides further simplification uses the Cholesky Factorization $LL^T = B$, which implies

$$\lambda LL^T x = Ax \Leftrightarrow \lambda L^T x = L^{-1}AL^{-T}L^T x \Leftrightarrow \bar{A}w = \lambda w \ , \ L^T x = w. \tag{9}$$

This reduction to the first order eigenvalue problem, when B is symmetric positive definite, is just one of the computational strategies available to solve Eq. (4). Other strategies for solving Eq. (4) include the QZ iteration (Golub, 1996), which is a generalization of the QR iteration method (Golub, 1996) for a pair of matrices (A, B), through reduction to a Hessenberg-triangular form (Varga, 1990).

Computational strategies to solve Eq. (3) without transformation to first order problem are usually applied for large and sparse linear second-order systems, and include the well-known Jacobi-Davidson method (Sleijpen et al, 1996; Zhou, 2006) and the SOAR method (Bai et al, 2005).

Well-known results on the sensitivity of problem given by Eq. (4) are given by several authors (Stuart, 1972. Eisenstat et al., 1998; Higham et al, 1999) including the well-known Bauer-Fike theorem (Golub, 1996). As for the original problem in Eq. (3), only a few but insightful results exist, among them we refer to (Sondipon, 1999), which relates relative changes in second order eigenvalues and eigenvectors with relative changes in the matrices M, C and K, in some matrix norm.

Since the matrices A and B of problem given by Eq. (4) are obtained by composing the matrices of problem (3) into double-sized matrices, and since no result on relationship between estimates for the two problems exist, one should expect to find situations in which Eq. (3) is far better conditioned than Eq. (4), possibly depending on which equation was used for the reduction.

A previous work (Carvalho, 2013) gathered computational evidences of this conjecture by benchmarking matrix datasets that showed that real eigenvalues obtained through reduction to Eq. (4) could yield to much better solutions of Eq. (3), obtained by using Krylov-subpace type techniques, real shifting, and by exploiting properties of the original system matrices. We propose here to extend those strategies, and to show how complex valued shifts and complex arithmetic strategies are able to refine any isolated solution of Eq. (3).

In section 2, we present our methodology and main definitions. In section 3, we present numerical results with public-domain test data comming from real-life vibrating structures.

2 Methods

2.1 Smallest singular value ratios

Let (λ_i, x_i), $i = 0, \ldots, 2n-1$ be exact values of eigenpairs of Eq. (3) and let μ and μ^* be approximations that are close to λ_i. Since any matrix $W_i = \lambda_i^2 M + \lambda_i C + K$ is singular, we will compare μ and μ^* on the grounds of how much singular they turn this matrix to be. Define the *nullity ratio* for a pair of approximations (μ, μ^*) by

$$N_R(\mu, \mu^*) = \frac{-\log_{10}(\min(\text{svd}((\mu^*)^2 M + \mu^* C + K)))}{-\log_{10}(\min(\text{svd}(\mu^2 M + \mu C + K)))} \tag{10}$$

where $\min(\text{svd}(U)))$ stands for the smallest singular value of a matrix U. If $N_R(\mu, \mu^*)$ is greater than one, than (μ^*) approximates some λ_i, $i = 1, ..., 2n - 1$, better than μ. For instance, for approximations μ and μ^* satisfying

$$\min(\text{svd}(\mu^2 M + \mu C + K)) = 1.0 \cdot 10^{-8}$$
$$\min(\text{svd}((\mu^*)^2 M + \mu^* C + K)) = 1.0 \cdot 10^{-10}$$

we have $N_R(\mu, \mu^*) = 10/8 = 1.25$.

Methodologically, given a set of matrices M, C and K, μ is obtained through reduction of Eq. (3) to Eq. (4), while, afterwards, μ^* is obtained through a refinement technique that uses such μ and Eq. (3).

2.2 The QZ method

One of the most used computational strategies for solving (4) is the QZ method. The pair of matrices (A, B) is reduced to a Hessemberg[1]-triangular pair (H, T) through orthogonal matrices U and V:

$$U^T A V = H \ , \ U^T B V = T \tag{11}$$

An iterative process, starting from matrices $H_0 = H$ and $T_0 = T$ and using QR decompositions, constructs a sequence of matrices (H_k, T_k) which, besides preserving the Hessemberg-triangular form, usually converges[2] to the generalized Real Schur decomposition (Λ, Γ) of the pair of matrices (A, B), from where real and complex eigenvalues can easily be obtained. More details can be found in (Golub, 1996).

2.3 Fundamental solutions and the Second-order Arnoldi method

Consider the second order eigenvalue problem given by Eq. (3) where matrix M is non-singular. Following (Claeyssen, 1990), there are two *matrix solutions* that can be defined by means of an associated second order difference problems. The *impulsive* one is defined by

$$\begin{cases} MH_{k+2} + CH_{k+1} + KH_k = 0, & k = 0, 1, 2, 3, \ldots \\ H_0 = 0, MH_1 = I. \end{cases} \tag{12}$$

where I denotes the identity $n \times n$ matrix. The classic one is defined by

$$\begin{cases} ML_{k+2} + CL_{k+1} + KL_k = 0, & k = 0, 1, 2, 3, \ldots \\ L_0 = I, MH_1 = 0. \end{cases} \tag{13}$$

Let p be a fixed positive integer. For any given pair of matrices $V_0 \in \mathbb{R}^{n \times p}$ and $U_0 \in \mathbb{R}^{n \times p}$, we define

$$Y_k = L_k U_0 + H_k V_0 \tag{14}$$

where H_k and L_k satisfy (12) and (13), respectively.

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[1]a Hessemberg matrix is a triangular matrix with an additional subdiagonal
[2]numerical analysis' strategies, like shifting, are applied to speed-up this convergence

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It is straightforward to verify that Y_k satisfies the recursive formula

$$\begin{cases} MY_{k+2} + CY_{k+1} + KY_k = 0, \quad k = 0, 1, 2, 3, \dots \\ Y_0 = U_0, MY_1 = V_0. \end{cases} \tag{15}$$

of matrices $\{Y_k\} \in \mathbb{R}^{n \times p}$ spanning Krylov-type like spaces from which approximations to dominant and isolated sets[3] of eigenvectors can be computed.

This strategy is closely related to the Second-order Arnoldi method described in (Bai et al, 2005). Furthermore, it is shown in (Carvalho, 2011) that, in order to construct a sequence $\{\alpha_k\}$ converging to a real single isolated eigenvalue λ_i that is the closest to a given real number σ, one can shift and invert (15) in order to derive the recurrence formulas

$$K_\sigma y_{k+2} + C_\sigma y_{k+1} + M y_k = 0 \tag{16}$$

$$\eta_k = |y_{k+1}|_2 \,, \; y_{k+2} \leftarrow \frac{y_{k+2}}{\eta_k} \,, \; y_{k+1} \leftarrow \frac{y_{k+1}}{\eta_k} \,, \; y_k \leftarrow \frac{y_k}{\eta_k} \tag{17}$$

$$\alpha_k = \frac{1}{y_{k+1}^T y_{k+2}} + \sigma \tag{18}$$

for any starting vectors $y_0, y_1 \in \mathbb{R}^n$, and for parametric matrices

$$C_\sigma = C + 2\sigma M \,, \; K_\sigma = K + \sigma C + \sigma^2 M. \tag{19}$$

We observe that the recurrence above can indeed be carried out for complex σ. In order to accomplish that,

$$\alpha_k = \frac{1}{\overline{y_{k+1}}^T y_{k+2}} + \sigma \tag{20}$$

must replace (18) in computational environments that work with complex valued numbers, vectors and matrices. We remark that $\overline{z}$ stands for the complex conjugate of z. There are a few resources to do that, and among them, there are the software libraries BLAS and LAPACK (Anderson et al, 1999).

On the other hand, if one wants to derive real valued formulas to implement this recurrence using only real numbers, vectors and matrices, we suggest to following approach: write $\sigma = \phi + \imath\psi$ for real ϕ and ψ and observe that complex valued matrices C_σ and K_σ in (19) can be written

$$C_\sigma = D_\sigma + \imath E_\sigma \,, \; K_\sigma = F_\sigma + \imath G_\sigma \tag{21}$$

for real valued matrices D_σ, E_σ, F_σ and G_σ satisfying

$$D_\sigma = C + 2\phi M \,, \; E_\sigma = 2\psi M \tag{22}$$

$$F_\sigma = (\phi^2 - \psi^2)M + \phi C + K \,, \; G_\sigma = 2\phi\psi M + \psi C. \tag{23}$$

If we also write $\overline{y_{k+1}}^T y_{k+2} = p_k + \imath q_k$ then (16)-(18) are equivalent to

$$\begin{bmatrix} F_\sigma & -G_\sigma \\ G_\sigma & F_\sigma \end{bmatrix} \begin{bmatrix} u_{k+2} \\ v_{k+2} \end{bmatrix} = - \begin{bmatrix} D_\sigma & -E_\sigma \\ E_\sigma & D_\sigma \end{bmatrix} \begin{bmatrix} u_{k+1} \\ v_{k+1} \end{bmatrix} - M \begin{bmatrix} u_k \\ v_k \end{bmatrix} \tag{24}$$

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[3]must have size p and must be self-conjugated

$$\eta_k = \sqrt{|u_{k+1}|_2^2 + |v_{k+1}|_2^2} \ , \ u_{k+2} \leftarrow \frac{u_{k+2}}{\eta_k} \ , \ v_{k+2} \leftarrow \frac{v_{k+2}}{\eta_k} \tag{25}$$

$$u_{k+1} \leftarrow \frac{u_{k+1}}{\eta_k} \ , \ v_{k+1} \leftarrow \frac{v_{k+1}}{\eta_k} \ , \ u_k \leftarrow \frac{u_k}{\eta_k} \ , \ v_k \leftarrow \frac{v_k}{\eta_k} \tag{26}$$

$$p_k \leftarrow u_{k+1}^T u_{k+2} + v_{k+1}^T v_{k+2} \ , \ q_k \leftarrow u_{k+1}^T v_{k+2} - v_{k+1}^T u_{k+2} \tag{27}$$

$$\alpha_k = \frac{p_k}{p_k^2 + q_k^2} \ , \ \beta_k = \frac{-q_k}{p_k^2 + q_k^2} \tag{28}$$

and $\{\alpha_k + \imath\beta_k\}$ converges to the eigenvalue λ_i which the closest to given $\sigma = \phi + \imath\psi$.

Unfortunately in Eq. (24) symmetry and positivity properties of the working matrices are lost. Therefore, depending on computational aspects like size and sparsity of the system matrices, working with equations (16),(17) and (20), that is, with complex valued matrices, can still be preferred because it can exploit symmetry, positiveness, sparsity and other properties the original system matrices might have.

In the computational results to be presented in next section, we have made an option to use complex valued numbers, vectors and matrices, with the aid of software libraries BLAS and LAPACK, instead of the complex-value free equations (21)-(28).

3   Results

3.1   Transmission tower with proportional damping

The data comes from the set BCSSTRUC1 of the *Harwell-Boeing* collection available in the *MatrixMarket* website (MatrixMarket). It regards a 153 d.o.f. transmission tower assumed to have proportional damping with $\alpha = 0.01$, $\beta = 0.0003$. These parameters were chosen so that the model has eigenvalue pairs sufficiently apart from each other, and then refinement technique proposed in last section could be applied successfully. This model has 153 complex conjugate eigenvalue pairs. Only eigenvalues with positive imaginary parts were refined. The experiments were done in an Intel Dual Core Pentium G630 2.7GHz Desktop under Ubuntu Linux and software *Matlab 2012b*, which is able to carry out double complex arithmetic operations with almost no adjust at all to the real data algorithms from (Carvalho, 2011).

Figures 1 and 2 show that the complex eigenvalues could be successfully improved through iterative refinement, yielding to *nullity ratios* varying from 1.1 to 2.8, regardless of the reduction formula that was used.

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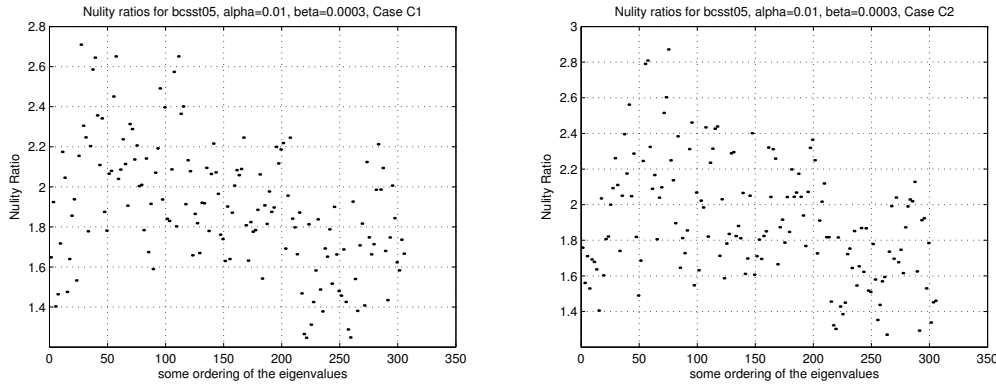
Figure 1: Nullity ratios for transmission tower, using formulas in Eq. (5) and (6).
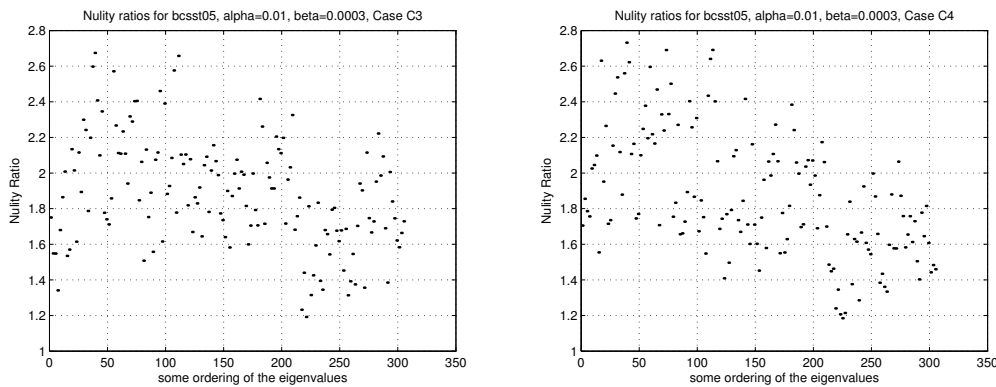

Figure 2: Nullity ratios for transmission tower, using formulas in Eq. (7) and (8).

3.2   Part of suspension bridge with proportional damping

The data comes from the set BCSSTRUC3 of the *Harwell-Boeing* collection (MatrixMarket). It regards a 817 d.o.f. part of a suspension bridge assumed to have proportional damping with the choices $\alpha = 0.1$, $\beta = 2.5 \cdot 10^{-5}$ so that all the 17 smallest magnitude eigenvalue pairs of the system are sufficiently isolated. The experiments were done in an Intel Dual Core Pentium G630 2.7GHz Desktop under Ubuntu Linux and Intel FORTRAN 90 compiler with *double complex* type data structures, with the aid of BLAS routines *zaxpy*, *zspmv*, *dznrm2* and *zscal*, and LAPACK routines *zsptrf* and *zsptrs* (Anderson et al, 1999).

Table 1 shows some outcomes in detail. It shows that the chosen subdominant set of frequencies was successfully refined with nullity ratios varying from 2 to 6.5. The singular-value computations for the nullity ratios were done in *Matlab 2012b*.

3.3   Half of engine inlet

The data comes from the set BCSSTRUC4 of the *Harwell-Boeing* collection (MatrixMarket). It regards a 1224 d.o.f. symmetric half of engine inlet from Boeing jetliner, under buckling analysis. Model is assumed to have proportional damping with the choices $\alpha = 10^{-2}$, $\beta = 10^{-4}$ so that the system has a set of eigenvalues which are more than 1% apart from each other (our strategy requires them to be isolated). A small subset of 458 eigeinvalues is taken; and only

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Table 1: Small subset of benchmark outcome for part of suspension bridge, using (6).

μ	μ^*	N_R
$-0.051084766291962 + 9.317209566176491i$	$-0.051085162062924 + 9.317207486083117i$	6.5
$-0.050031035971404 + 1.582320118444923i$	$-0.05003132798736 + 1.582319772644305i$	5.8
$-0.050026089882555 + 1.442364218095966i$	$-0.050026036465881 + 1.442364263353126i$	2.2
$-0.050144725178335 + 3.400309779358339i$	$-0.050144557825537 + 3.400310511918943i$	3.6
$-0.050114336761382 + 3.027179214317865i$	$-0.050114579072032 + 3.027179263425846i$	3.0
$-0.050208054884942 + 4.082572936190894i$	$-0.050208373965475 + 4.082572271850640i$	2.7
$-0.050217871526307 + 4.173761937451974i$	$-0.050217785112027 + 4.173761748830215i$	2.8
$-0.050259834443233 + 4.564440771379331i$	$-0.050260458057345 + 4.564440653355708i$	3.3
$-0.050301883707487 + 4.917847147751918i$	$-0.050302346854857 + 4.917846893261517i$	3.2
$-0.050403489581206 + 5.685709562415424i$	$-0.050404122889866 + 5.685709332472713i$	3.0
$-0.050433043547391 + 5.886671744926964i$	$-0.050433193101629 + 5.886671778144693i$	2.9
$-0.050505086337247 + 6.357520894181837i$	$-0.050505257787513 + 6.357520917769455i$	2.9
$-0.050893272757203 + 8.455165630217271i$	$-0.050893655173541 + 8.455165505136536i$	4.7
$-0.050626496684400 + 7.078031137471627i$	$-0.050626263496229 + 7.078030564842803i$	4.6
$-0.050778545465319 + 7.891432941597847i$	$-0.050778465926318 + 7.891431787273492i$	4.3
$-0.050807677914873 + 8.039202592368438i$	$-0.050807891968324 + 8.039202449512876i$	3.7
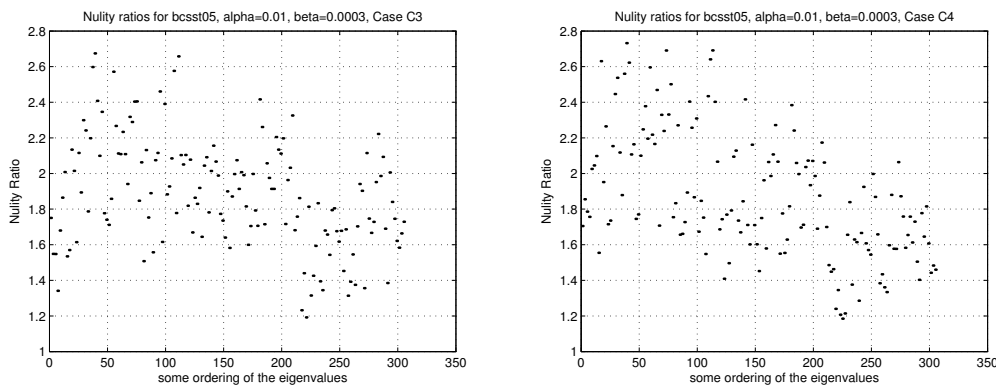
Figure 3: Nullity ratios for transmission tower, using formulas in Eq. (7) and (8).

one representative of each complex conjugated pair is chosen. Only formulas in Eq. (5), which are the most well-known, are considered. The experiments were done in an Intel Dual Core Pentium G630 2.7GHz Desktop under Ubuntu Linux and Intel FORTRAN 90 compiler with *double complex* type data structures, with the aid of BLAS routines *zaxpy*, *zspmv*, *dznrm2* and *zscal*, and LAPACK routines *zsptrf* and *zsptrs*. The singular-value computations for the nullity ratios were done in *Matlab 2012b*.
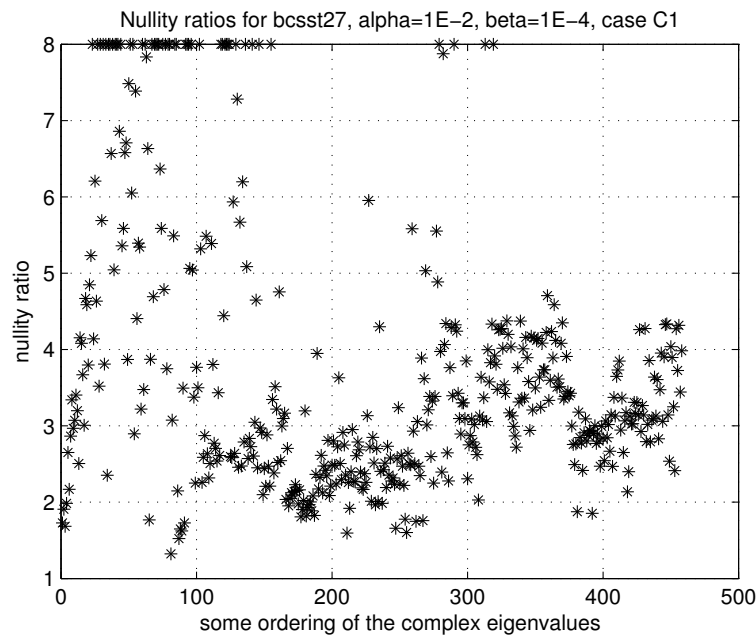
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Figure 4: **Nullity ratios for half of engine inlet, using formulas in Eq. (5) .**

Figure 4 shows that the complex eigenvalues could be successfully improved through iterative refinement, yielding to *nullity ratios* mostly varying from 1.5 to 4.0.

4 Conclusion

Krylov-subspace type iterations using shifts and inversion, already shown to be able to refine real isolated eigenvalues of second order linear systems, were adapted to refine also complex conjugate isolated pairs. The proposed technique uses the original system matrices directly, exploiting properties like their symmetry and positiveness, and taking advantage that, in its original form, the eigenvalue problem seams to be far better conditioned than the generalized first order eigenvalue problem to which it is usually reduced to. Benchmarks using test data in computational environments capable to carry out complex valued arithmetic computations are made. Computational evidence has shown that a class of strategies that are usually applied in the context of large and sparse linear structures can also be of great benefit for improving the solution of the eigenvalue problem for small and medium sized dense structures, through iterative refinement.

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