NUMERICAL INTEGRATION OF A SATELLITE ORBIT WITH KS TRANSFORMATION

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Abstract. *A satellite orbit is mainly influenced by central body gravitational forces. For a satellite in LEO (Low Earth Orbit), MEO (Medium Earth Orbit) or GEO (Geosynchronous Earth Orbit) the Earth´s gravity distribution and other perturbations determine the position and velocity changes in function of time. If the motion is around a spherical body with homogenous mass distribution and without perturbative forces, the orbit must be cyclic like the Two Body Problem (TBP) or Keplerian Orbit. Different numerical methods can be applied for solving the Ordinary Differential Equations (ODE´s). In this work a fourth-order fixed step-size Runge-Kutta numerical integrator (RK4) was implemented. With satellite´s position and velocity in inertial reference frame at zero time (orbit initial conditions) and solving the ODE´s with RK4 it is possible to know the satellite position and velocity at any time, with a certain level of accuracy. When the integration time is equal to the orbit period time, in a Keplerian orbit, the initial and final orbit data are compared to obtain the integration error in position and velocity. To better accuracy it is recommended to change the ODE´s from a Newtonian system by a time transformation to a stable Liapunov system and finally to Kunstaanheimo-Stiefel (KS) transformed system. In this paper the results obtained by applying the KS transformation to the orbit ODE´s, its accuracy and error analysis for different step-sizes of integration in a satellite orbit propagation are presented. Additionally, the orbits propagated are compared in terms of performance, CPU time and degradation of*

accuracy. Finally conclusions are drawn showing the beneficial aspects of using the KS transformation as an efficient technique for precise orbit integration of Earth satellites.

Keywords: *Satellite Orbit, Numerical Integration, Time Transformation, KS transformation.*

1 INTRODUCTION

In the orbit motion the equation that describes the trajectory of a satellite around the Earth is

$$\ddot{\boldsymbol{r}} = -\frac{\mu}{r^3}\boldsymbol{r} + \left(\boldsymbol{P} - \frac{\partial U}{\partial \boldsymbol{r}}\right) \quad . \tag{1}$$

where $\ddot{r}$ is the acceleration, $r = \sqrt{x^2 + y^2 + z^2}$ the position radius, μ the gravitational constant of central body, $\boldsymbol{P}$ external perturbations and U is the potential. If the central body is a homogenous mass sphere and the external perturbative forces are neglected the motion equation reduces to $\ddot{\boldsymbol{r}} = -\frac{\mu}{r^3}\boldsymbol{r}$, known as keplerian motion in a Newtonian system of differential equations (Bate, Mueller & White, 1971).

Solving the motion Eq. 1 by numerical integration methods, in this case by a 4th Order Runge-Kutta (RK4) with a fixed step-size, it is possible to obtain the six values for the body position and velocity in any time in the Cartesian system.

Because of the Kepler laws, in elliptical orbits the satellite is faster in the perigee and a fixed step-size integration can generate position and velocity errors that can be propagate along the trajectory. To solve this problem, a new independent variable is formulated together with a Sundman time transformation, where s represent the eccentric anomaly (Berry & Healy, 2002):

$$dt = r ds, \tag{2}$$

where s represent an “Eccentric anomaly” like variable. In such transformation s is called fictitious time, given by:

$$t' = r . \tag{3}$$

The motion equation in the time transformed system is:

$$\boldsymbol{r''} = \frac{r'}{r}\boldsymbol{r'} - \frac{\mu}{r}\boldsymbol{r} + r^2\left(\boldsymbol{P} - \frac{\partial U}{\partial \boldsymbol{r}}\right) \quad . \tag{4}$$

The velocity magnitude is given by $v^2 = \frac{1}{r^2}|\boldsymbol{r'}|^2$ with $r' = dr/ds$. The system can be written in function of the orbital energy with a new differential equation (Pellegrini, Russel & Vittaldev, 2013). The orbit mechanical energy $H = \frac{\mu}{r} - \frac{1}{2r^2}|r'|^2$, and its differential equation in function of fictitious time is:

$$H' = -\boldsymbol{P} \bullet \boldsymbol{r'} \quad . \tag{5}$$

The energy constraint can be inserted in the motion equations and this give rise to equations in a stabilized form:

$$\boldsymbol{r''} = \frac{r'}{r}\boldsymbol{r'} - \left(\frac{1}{2r^2}\left|r'\right|^2 + U + H\right)\boldsymbol{r} + r^2\left(\boldsymbol{P} - \frac{\partial U}{\partial \boldsymbol{r}}\right) \quad . \tag{6}$$

The stabilized system has eight ODE´s, time, energy, 3 components of position and 3 for velocity (Yam, Izzo & Biscani, 2010).

Another method for orbit propagation is the Kustaanheimo-Stiefel (KS) transformation. This system transform the three dimensional orbital system into a 4 dimensional. The aim of this transformation is to obtain a harmonic oscillator system, with is both stabilized and regularized. The new system has 10 EDO's. The KS transformation matrix ***L(u)*** are presented as:

$$\boldsymbol{L(u)} = \begin{bmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_1 \\ u_4 & -u_3 & u_2 & -u_1 \end{bmatrix} \quad . \tag{7}$$

The transformation from 4-D to 3-D system is

$$\boldsymbol{r} = \boldsymbol{L(u)u} \quad . \tag{8}$$

The new energy variation and motion equation are finally:

$$H' = -2\left(\boldsymbol{L^T(u)P} \bullet \boldsymbol{u'}\right) \quad . \tag{9}$$

$$\boldsymbol{u''} = -\frac{(H+U)}{2}\boldsymbol{u} + \frac{\left|\boldsymbol{u}\right|^2}{2}\left(-\frac{1}{2}\frac{\partial U}{\partial \boldsymbol{u}} + L^T(\boldsymbol{u})\boldsymbol{P}\right) \quad . \tag{10}$$

The development of KS equations system (Eq. 7 to Eq. 10) are presented in Stiefel & Scheifele (1971), Bond (1974) and Neto (1974). Another studies and set of equations for gravity perturbed orbit with KS transformation are proposed in Portilla (1996), Sharaf and Selim (2013).

2 METHODOLOGY

In order to illustrate the powerfulness of such transformations, a problem to solve is the elliptical orbit for a satellite around the Earth without perturbing forces. For solving the EDO's a RK4 integrator is implemented in the numerical orbit propagator.

The orbit was propagated and the results obtained with three methods were compared: conventional Newtonian, time transformation and KS transformation (called regularized system). The number of integration steps was incremented to better accuracy. The error is calculated for velocity and position components. The table 1 and table 2 show the initial conditions for the corresponding position and velocity vectors and the orbital elements, respectively.

Table 1. Orbit initial conditions

Coordinate	Position (m)	Velocity (m/s)
X	1888980.4103698	-9585.79511076297
Y	6652209.67475597	2413.57051166562
Z	902482.883545056	2273.50403709003

Table 2. Orbital elements

Orbital element	Value
Semi major axis	34869261m
Eccentricity	0.8
Inclination	15°
RAAN	45°
AOP	30°

3 RESULTS

For the satellite orbit propagation of the Keplerian motion (TBP) are selected three methods: Conventional method (called Cowell´s method), time transformation and KS transformation. The results and orbit geometry for one period propagation applying the three methods are shows in Fig. 1. In Cowell´s method, the orbital period (T) is divided by the step-size to obtain the time interval for the propagation, in Sundman and KS methods the eccentric anomaly (S) is divided by the integration step.

The results obtained from the propagation of 18 orbits, applying the three methods and variation the step-size are presented in Fig. 2, Fig. 3 and Fig. 4. The numerical codes were written in FORTRAN. A CPU with Core i7 2.4GHz processor and 12GB-RAM memory was implemented to propagate the orbits.

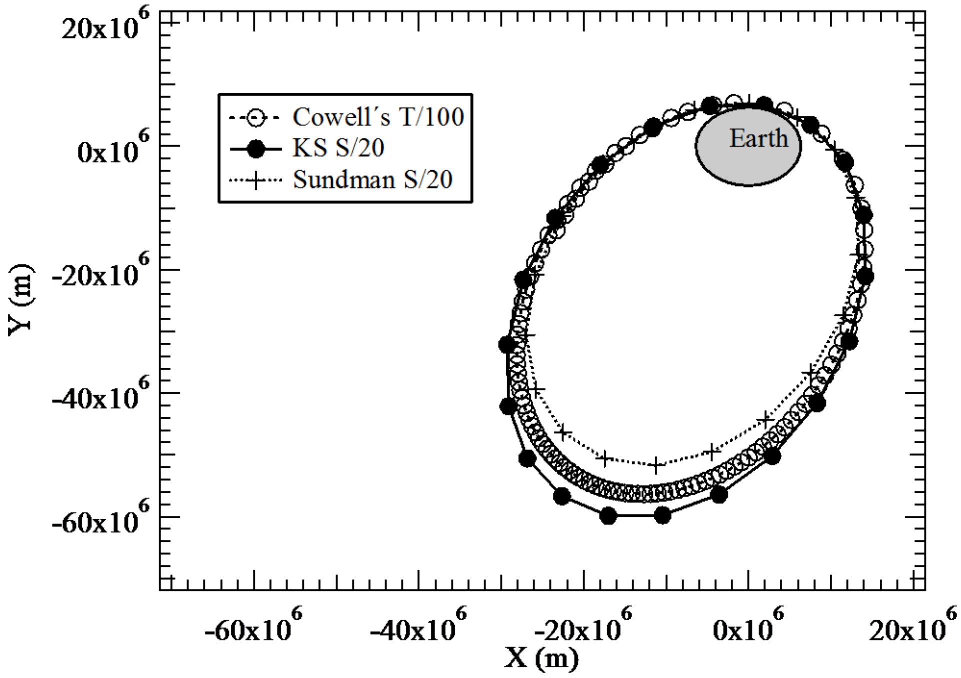

Figure 1. One orbit propagation in XY-plane from three methods.

Errors in position and velocity vectors were calculated, Fig. 2 and Fig. 3 shows the results. The results from KS transformation present highest accuracy compared to Cowell´s method. Errors lower than 1.5x10^{-11} % and 4 x10^{-13} % were calculated for position vector and velocity vector applying the KS transformation along one orbit in two continuous perigees.

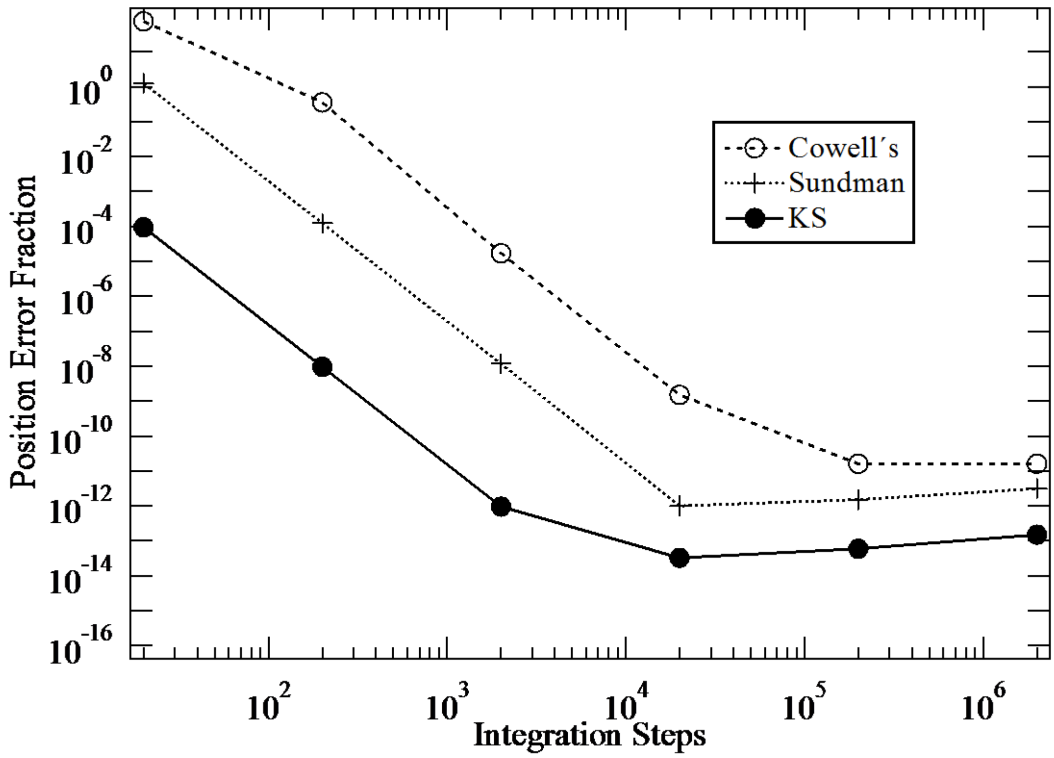

Figure 2. Position Error.

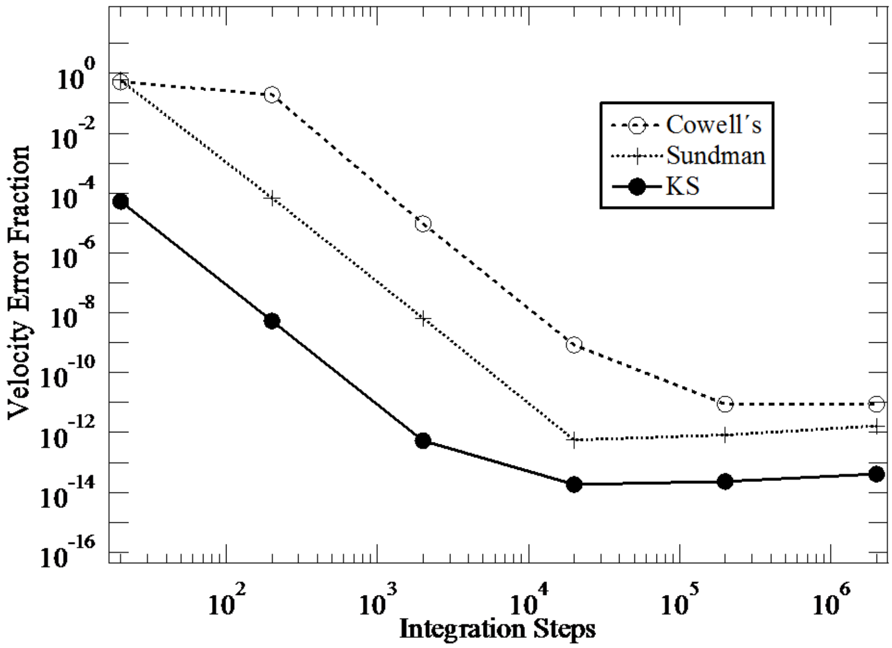

Figure 3. Velocity Error.

The Fig. 4 presents the CPU time in function of integration steps. It is possible to see that the Cowell´s method generates a highest computational cost, and for small integration steps the KS transformation is faster than Sundman transformation.

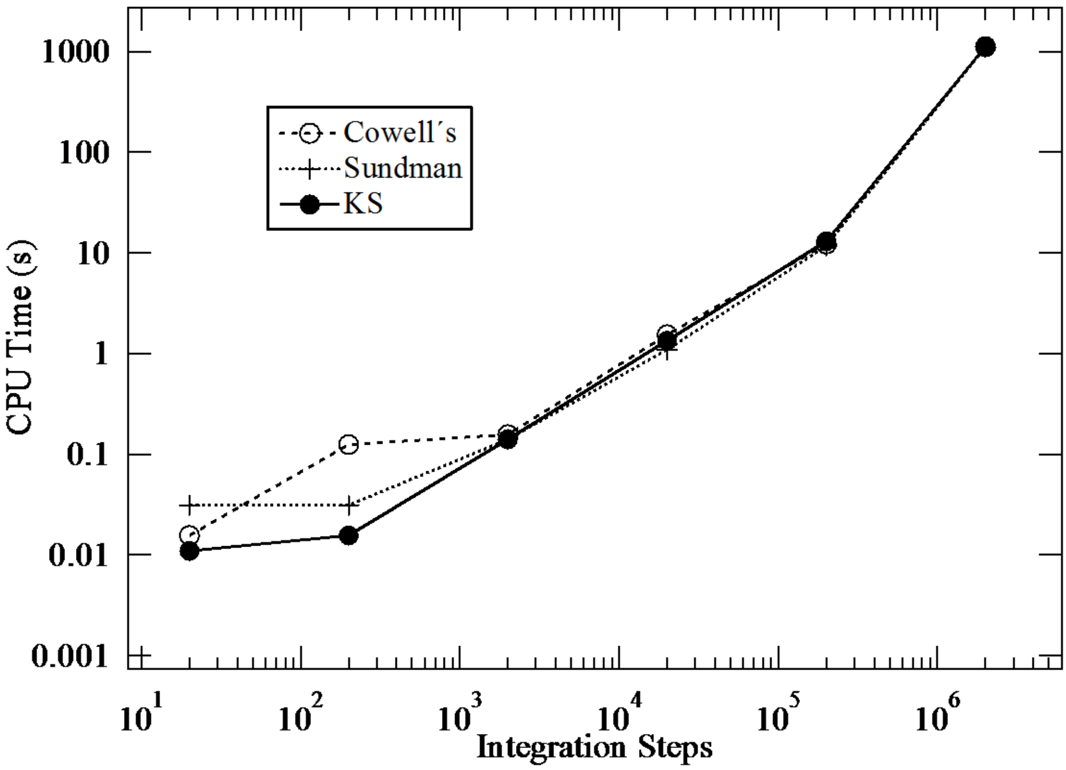

Figure 4. CPU Time.

4 CONCLUSIONS

Three propagator methods were implemented to describe the motion of a satellite around the Earth into a high eccentricity orbit for one orbital period. The FORTRAN code has a RK4 for the numerical integration. The KS transformation presented better accuracy for velocity and position vectors. Compared to Sundman transformation and Cowell´s method, the KS transformation is more stable and accurate for small integration steps (S/20) and have a lower CPU cost. Cowell´s method present uncertainties in position for integration steps lowers than T/200, thus it is not recommended the implementation of this method for orbit propagation. In high eccentricity orbits the propagation with the KS transformation reduce the errors, but, in low eccentricity orbits like circular ones, the three methods present the same integration points because the time coincides with the angular or S-step.

KS and Sundman propagations with step-size around 10^4 have better computational cost and lowest errors than Cowell´s method. It is possible to obtain betters results incrementing the step-size in Cowell´s method but at the same time has a CPU burden augmentation.

Futures studies are proposed to implement the geopotential harmonic J2 in orbit propagation for comparison between Keplerian orbit and perturbed orbit.

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