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APPROXIMATED SOLUTION OF LINEAR SYSTEMS ARISING FROM TOPOLOGY OPTIMIZATION OF STRUCTURES

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Abstract. *A common problem in topology optimization is the minimization of the compliance of a static structure, subject to a volume constraint. No matter what method is used to solve this problem, the computational cost is dominated by the solution of the system of equations associated with the equilibrium conditions. If the structure has an elastic material and is subjected to small displacements, one linear system need to be solved at each iteration of the optimization algorithm, in order to obtain the nodal displacements, which are used to evaluate the objective function. Since the stiffness matrix of the structure is symmetric and positive definite, this linear system is frequently solved using the Cholesky factorization, although this method can be expensive for large-scale problems. To overcome this difficulty, Amir, Bendsøe & Sigmund (2009) presented a strategy based on the combined approximations approach, proposed by Kirsch (1991), which consists basically in reusing the Cholesky factorization. In this work, we combine this approach with the Sequential Piecewise Linear Programming method, proposed by Gomes & Senne (2014), and discuss the reduction of the computational cost of the solution of the linear systems when this strategy is applied to a classical problem.*

Keywords: *Topology optimization, Linear systems, Approximated solutions*

1 INTRODUCTION

The simplest topology optimization problem is the minimization of the compliance of a structure. The objective is to find the stiffest structure that fits into the domain, satisfies the boundary conditions and has a prescribed volume. After domain discretization, this problem becomes

$$\begin{aligned} \min_{\rho} \quad & \mathbf{f}^T\mathbf{u} \\ \text{s. t.} \quad & \mathbf{K}(\rho)\mathbf{u} = \mathbf{f} \\ & \sum_{i=1}^{n} v_i\,\rho_i \leq V^* \\ & \rho_{\min} \leq \rho_i \leq 1\,, \qquad i = 1,\,\ldots,\,n\,, \end{aligned} \tag{1}$$

where n is the number of elements of the domain, ρ_i and v_i are, respectively, the density and the volume of the i-th element, V^* is the upper bound for the volume of the structure, $\mathbf{f}$ is the vector of nodal forces associated to the external loads, $\mathbf{K}(\rho)$ and $\mathbf{u}$ are the global stiffness matrix of the structure and the vector of nodal displacements, respectively, and $\rho_{\min} > 0$ is a parameter that represents a minimum allowed density to avoid the singularity of the global stiffness matrix.

When the SIMP model (Bendsøe, 1989) is used to avoid intermediate densities, the global stiffness matrix is given by

$$\mathbf{K}(\rho) = \sum_{i=1}^{n} \rho_i^p \mathbf{K}_i\,, \tag{2}$$

where $p \geq 1$ is a penalty parameter and $\mathbf{K}_i$ is the local stiffness matrix of the i-th element.

To evaluate the objective function of the topology optimization given by Eq. (1), we need to compute the nodal displacements vector $\mathbf{u}$ by solving the linear system

$$\mathbf{K}(\rho)\mathbf{u} = \mathbf{f}\,, \tag{3}$$

that represents the static equilibrium conditions of the structure. The global stiffness matrix $\mathbf{K}(\rho)$ is symmetric, and, if $\rho_i \geq \rho_{\min}$ for all $i = 1,\,\ldots,\,n$, $\mathbf{K}(\rho)$ is nonsingular. In this case, from Eq. (3), we obtain

$$\mathbf{u} = \mathbf{K}(\rho)^{-1}\mathbf{f}\,,$$

and we can rewrite the problem in Eq. (1) as

$$\begin{aligned} \min_{\rho} \quad & \mathbf{f}^T\mathbf{K}(\rho)^{-1}\mathbf{f} \\ \text{s. t.} \quad & \sum_{i=1}^{n} v_i\,\rho_i \leq V \\ & \rho_{\min} \leq \rho_i \leq 1\,, \qquad i = 1,\,\ldots,\,n\,. \end{aligned}$$

After imposing the boundary conditions of the structure, $\mathbf{K}(\rho)$ becomes definite positive. Thus, in this case, we usually solve the linear system in Eq. (3) through the Cholesky factorization. The most expensive step in the solution of the topology optimization problem given by Eq. (1) is the computation of the Cholesky factor of the global stiffness matrix $\mathbf{K} \equiv \mathbf{K}(\rho)$. Having this in mind, in this work, our focus is to investigate a way to reduce the computational effort involving the solution of the linear system in Eq. (3). One possible way is to reuse the Cholesky

factorization of $\mathbf{K}_0 \equiv \mathbf{K}_0(\rho)$ evaluated at some iteration m_0 of an optimization algorithm in the subsequent iterations $m_0+1, \ldots, m_0+r$, for some positive integer r . So, reusing the Cholesky factorization of $\mathbf{K}_0$, we will find an approximated solution $\widetilde{\mathbf{u}}$ for the linear system (3) . This approximated solution is obtained through an approach known as *combined approximations*, originally proposed by Kirsch (1991) and adapted to topology optimization problems by Amir, Bendsøe & Sigmund (2009). Inspired by their work, we will describe this approach in the next section, for completeness of the text.

2 COMBINED APPROXIMATIONS APPROACH

Let $\mathbf{K}_0$ be the global stifness matrix obtained at an iteration m_0 of an optimization algorithm, and its Cholesky factorization $\mathbf{K}_0 = \mathbf{G}_0^T\mathbf{G}_0$, where $\mathbf{G}_0$ is an upper triangular matrix. Fixed an iteration $m > m_0$, we can write the global stiffness matrix $\mathbf{K}$ evaluated at this iteration as

$$\mathbf{K} = \mathbf{K}_0 + \Delta\mathbf{K} ,$$

where

$$\Delta\mathbf{K} \equiv \Delta\mathbf{K}(\rho, \rho^{(0)}) = \mathbf{K}(\rho) - \mathbf{K}_0(\rho^{(0)}) = \textstyle\sum_{i=1}^{n}[\rho_i^p - (\rho_i^{(0)})^p]\,\mathbf{K}_i ,$$

represents the matrix of changes in stiffness due to changes in the values of the design variables. So, we can rewrite Eq. (3) as

$$(\mathbf{K}_0 + \Delta\mathbf{K})\mathbf{u} = \mathbf{f} . \tag{4}$$

Having Eq. (4) in hands, we define at the current iteration m the following recurrence relation

$$\mathbf{K}_0\mathbf{u}^{(k)} = \mathbf{f} - (\Delta\,\mathbf{K})\mathbf{u}^{(k-1)} , \qquad k = 1, 2, \ldots . \tag{5}$$

From Eq. (5), we observe that

$$\mathbf{u}^{(k)} = \mathbf{K}_0^{-1}\mathbf{f} - \mathbf{K}_0^{-1}(\Delta\mathbf{K})\mathbf{u}^{(k-1)} , \qquad k = 1, 2, \ldots . \tag{6}$$

Defining

$$\mathbf{u}^{(1)} = \mathbf{K}_0^{-1}\,\mathbf{f}$$

and

$$\mathbf{B} = \mathbf{K}_0^{-1}\Delta\mathbf{K} ,$$

we obtain, from Eq. (6),

$$\mathbf{u}^{(k)} = \mathbf{u}^{(1)} - \mathbf{B}\mathbf{u}^{(k-1)} , \qquad k = 1, 2, \ldots . \tag{7}$$

Note that, from Eq. (7), we have

$$\begin{aligned}
\mathbf{u}^{(2)} &= (\mathbf{I} - \mathbf{B})\,\mathbf{u}^{(1)} , \\
\mathbf{u}^{(3)} &= (\mathbf{I} - \mathbf{B} + \mathbf{B}^2)\,\mathbf{u}^{(1)} , \\
\mathbf{u}^{(4)} &= (\mathbf{I} - \mathbf{B} + \mathbf{B}^2 - \mathbf{B}^3)\,\mathbf{u}^{(1)} , \\
&\vdots \\
\mathbf{u}^{(k)} &= (\mathbf{I} - \mathbf{B} + \mathbf{B}^2 - \mathbf{B}^3 + \cdots + (-1)^{k-1}\mathbf{B}^{k-1})\,\mathbf{u}^{(1)} .
\end{aligned} \tag{8}$$

It can be shown that, if $\lim_{k\to+\infty} \mathbf{B}^k = \mathbf{0}$, then $\mathbf{I}+\mathbf{B}$ is nonsingular, and

$$(\mathbf{I}+\mathbf{B})^{-1} = \mathbf{I}-\mathbf{B}+\mathbf{B}^2-\mathbf{B}^3+\cdots+(-1)^k\mathbf{B}^k+\cdots .$$

Thus, in this case,

$$\mathbf{u} = (\mathbf{I}-\mathbf{B}+\mathbf{B}^2-\mathbf{B}^3+\cdots+(-1)^k\mathbf{B}^k+\cdots)\mathbf{u}^{(1)} = (\mathbf{I}+\mathbf{B})^{-1}\mathbf{u}^{(1)}$$

is the solution of the original linear system in Eq. (3). In fact,

$$\begin{aligned}
\mathbf{K}\mathbf{u} &= \mathbf{K}(\mathbf{I}+\mathbf{B})^{-1}\mathbf{u}^1 \\
&= \mathbf{K}(\mathbf{I}+\mathbf{B})^{-1}\mathbf{K}_0^{-1}\mathbf{f} \\
&= (\mathbf{K}_0+\Delta\mathbf{K})(\mathbf{I}+\mathbf{B})^{-1}\mathbf{K}_0^{-1}\mathbf{f} \\
&= (\mathbf{K}_0+\Delta\mathbf{K})(\mathbf{K}_0^{-1}\mathbf{K}_0+\mathbf{K}_0^{-1}\Delta\mathbf{K})^{-1}\mathbf{K}_0^{-1}\mathbf{f} \\
&= (\mathbf{K}_0+\Delta\mathbf{K})[\mathbf{K}_0^{-1}(\mathbf{K}_0+\Delta\mathbf{K})]^{-1}\mathbf{K}_0^{-1}\mathbf{f} \\
&= (\mathbf{K}_0+\Delta\mathbf{K})(\mathbf{K}_0+\Delta\mathbf{K})^{-1}\mathbf{K}_0\mathbf{K}_0^{-1}\mathbf{f} \\
&= \mathbf{f} .
\end{aligned}$$

The approximated solution $\tilde{\mathbf{u}}$ of the linear system in Eq. (3) is expressed as a linear combination of the s first terms of the vectors in Eq. (8), obtained from the recurrence relation in Eq. (7):

$$\tilde{\mathbf{u}} = \alpha_1\mathbf{u}^{(1)} + \alpha_2\mathbf{u}^{(2)} + \ldots + \alpha_s\mathbf{u}^{(s)} . \tag{9}$$

In general, s must to be a small positive integer number (for example, $s \leq 10$).

Substituting the expressions in Eq. (8) into Eq. (9), we obtain

$$\begin{aligned}
\tilde{\mathbf{u}} &= \alpha_1\mathbf{u}^{(1)} + \alpha_2(\mathbf{I}-\mathbf{B})\mathbf{u}^{(1)} + \alpha_3(\mathbf{I}-\mathbf{B}+\mathbf{B}^2)\mathbf{u}^{(1)} + \cdots + \\
&\quad + \alpha_s(\mathbf{I}-\mathbf{B}+\mathbf{B}^2-\mathbf{B}^3+\cdots+(-1)^{s-1}\mathbf{B}^{s-1})\mathbf{u}^{(1)} .
\end{aligned} \tag{10}$$

Rearranging the terms in Eq. (10), we have

$$\begin{aligned}
\tilde{\mathbf{u}} &= (\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_s)\mathbf{u}^{(1)} + (\alpha_2+\alpha_3+\cdots+\alpha_s)(-\mathbf{B}\mathbf{u}^{(1)}) + \\
&\quad + (\alpha_3+\cdots+\alpha_s)(\mathbf{B}^2\mathbf{u}^{(1)}) + \alpha_s((-1)^{s-1}\mathbf{B}^{s-1}\mathbf{u}^{(1)}) .
\end{aligned}$$

Defining

$$\begin{aligned}
\hat{\mathbf{u}}^{(1)} &= \mathbf{u}^{(1)} , \\
\hat{\mathbf{u}}^{(2)} &= -\mathbf{B}\mathbf{u}^{(1)} = -\mathbf{B}\hat{\mathbf{u}}^{(1)} \\
\hat{\mathbf{u}}^{(3)} &= \mathbf{B}^2\mathbf{u}^{(1)} = \mathbf{B}^2\hat{\mathbf{u}}^{(1)} = -\mathbf{B}\hat{\mathbf{u}}^{(2)} , \\
\hat{\mathbf{u}}^{(4)} &= -\mathbf{B}^3\mathbf{u}^{(1)} = -\mathbf{B}^3\hat{\mathbf{u}}^{(1)} = -\mathbf{B}\hat{\mathbf{u}}^{(3)} , \\
&\vdots \\
\hat{\mathbf{u}}^{(s)} &= -\mathbf{B}\hat{\mathbf{u}}^{(s-1)}
\end{aligned} \tag{11}$$

and

$$
\begin{aligned}
y_1 &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \cdots + \alpha_s \\
y_2 &= \alpha_2 + \alpha_3 + \alpha_4 + \cdots + \alpha_s \\
y_3 &= \alpha_3 + \alpha_4 + \cdots + \alpha_s \\
y_4 &= \alpha_4 + \cdots + \alpha_s \\
&\vdots \\
y_s &= \alpha_s \,,
\end{aligned}
$$

we find

$$\widetilde{\mathbf{u}} \,=\, y_1\widehat{\mathbf{u}}^{(1)} \,+\, y_2\widehat{\mathbf{u}}^{(2)} \,+\, \ldots \,+\, y_s\widehat{\mathbf{u}}^{(s)} \,=\, \mathbf{R}_B\,\mathbf{y}\,, \tag{12}$$

where

$$\mathbf{R}_B \,=\, [\widehat{\mathbf{u}}^{(1)}\ \widehat{\mathbf{u}}^{(2)}\ \ldots\ \widehat{\mathbf{u}}^{(s)}] \tag{13}$$

and

$$\mathbf{y} = [y_1\ \ldots\ y_s]^T\,.$$

Substituting the approximated solution $\widetilde{\mathbf{u}}$ defined by Eq. (12) into Eq. (3), we get

$$\mathbf{K}\mathbf{R}_B\,\mathbf{y} = \mathbf{f}\,. \tag{14}$$

Now, multiplying Eq. (14) by $\mathbf{R}_B^T$ on the left, we have

$$\mathbf{R}_B^T\mathbf{K}\mathbf{R}_B\,\mathbf{y} = \mathbf{R}_B^T\mathbf{f}\,. \tag{15}$$

Denoting

$$\mathbf{K}_R = \mathbf{R}_B^T\mathbf{K}\mathbf{R}_B \tag{16}$$

and

$$\mathbf{f}_R = \mathbf{R}_B^T\mathbf{f}\,,$$

we rewrite Eq. (15) as

$$\mathbf{K}_R\mathbf{y} = \mathbf{f}_R\,. \tag{17}$$

Thus, instead of solving the large linear system in Eq. (3), we solve a small one defined by Eq. (17), which gives us an approximated solution to the original linear system of Eq. (3). To obtain a more stable numerical scheme, we use the $\mathbf{QR}$ factorization to solve the linear system given by Eq. (17).

Using the approach of the combined approximations described above, the topology optimization problem in Eq. (1) becomes

$$
\begin{aligned}
\min_{\rho}\ & \mathbf{y}^T\mathbf{R}_B^T\mathbf{K}\mathbf{R}_B\mathbf{y} \\
\text{s. t.}\ & \mathbf{R}_B^T\mathbf{K}\mathbf{R}_B\mathbf{y} \,=\, \mathbf{R}_B^T\mathbf{f} \\
& \sum_{i=1}^{n} v_i\,\rho_i \le V^* \\
& \rho_{\min} \le \rho_i \le 1\,, \qquad i = 1,\,\ldots,\,n\,.
\end{aligned}
\tag{18}
$$

It is possible to solve the topology optimization problems in Eq. (1) and in Eq. (18) adopting several methods, as, for example, the Method of Moving Asymptotes (MMA, introduced by Svanberg, 1987), the globally convergent version of the Sequential Linear Programming (SLP, presented by Gomes & Senne, 2011), and the Sequential Piecewise Linear Programming (SPLP, introduced by Gomes & Senne, 2014).

3 NUMERICAL TESTS

In this work, with the goal of reducing the computational effort of evaluating the objective function of the topology optimization problem in Eq. (1), we propose an algorithm that joins the SPLP method (Gomes & Senne, 2014) with the combined approximations approach (described in Section 2).

3.1 Implementation details

Now, we will present some features concerning the pratical implementation of the algorithm.

To avoid the checkerboard-like pattern in the optimal structure (see Díaz & Sigmund, 1995), we adopt the density filter proposed by Bruns and Tortorelli (2003). With the aim of guaranteeing that the global stiffness matrix $\mathbf{K}$ in Eq. (2) will be nonsingular, we choose $\rho_{min} = 10^{-3}$ as the minimum value allowed for the densities in Eq. (1).

In order to avoid local minima, the optimal solutions were obtained through a gradative increasing of the penalty parameter p of the SIMP method, from 1 to 3, in steps of 1. This strategy, known as *continuation method*, was proposed by Allaire and Francfort (1993).

Solution of the linear systems. The number s of vectors in Eq. (11) plays a important role in the efficiency of combined approximations approach to obtain the vector of nodal displacements $\widetilde{\mathbf{u}}$, that is the approximated solution of the linear system in Eq. (3). Amir, Bendsøe & Sigmund (2009) suggested a criterion for choosing such a number based on the value of the relative magnitude of the residual forces

$$\frac{\|\mathbf{f} - \mathbf{K}\mathbf{R}_B\mathbf{y}\|_2}{\|\mathbf{f}\|_2}, \tag{19}$$

(where $\|\cdot\|_2$ denotes the Euclidean norm) and they established a minimum and a maximum number of vectors generated (denoted, respectively, by $s_{\min}$ and $s_{\max}$). Having this in mind, they presented the following procedure to evaluate $\widetilde{\mathbf{u}}$ in Eq. (12). First of all, compute $s = s_{\min}$ vectors using Eq. (11) to obtain $\mathbf{R}_B$ in Eq. (13). After this, solve the linear system in Eq. (17), and use its solution $\mathbf{y}$ to obtain $\widetilde{\mathbf{u}}$ in Eq. (12). Then, compute the relative magnitude of the residual forces given in Eq. (19). If its value is less than 10^{-2}, we consider that $\widetilde{\mathbf{u}}$ is a good approximation to the true solution $\mathbf{u}$ of the original linear system in Eq. (3), and we stop this process. Otherwise, generate the $(s+1)$-th term of the sequence in Eq. (11), put it aside to the s-th column of the matrix $\mathbf{R}_B$, solve again the linear system in Eq. (17) to obtain a new approximation $\widetilde{\mathbf{u}}$, and recompute the relative magnitude of the residual forces in Eq. (19). Repeat all this process until the relative measure becomes less than 10^{-2} or until we reach $s = s_{\max}$.

In their numerical tests, Amir, Bendsøe & Sigmund (2009) took $s_{\min} = 1$ and $s_{\max} = 4$, and they update the Cholesky decomposition of $\mathbf{K}$ under two possible conditions: after a

fixed number of consecutive iterations of the optimization algorithm, or when the current vector densities $\rho^{(k)}$ is significantly different from the vector $\rho^{(0)}$ corresponding to the last factorized matrix. But, in order to investigate the efficiency of the combined approximations strategy with a larger number of vectors, we intially adopted $s_{\min} = 3$ and $s_{\max} = 25$. We observed that, at the initial iterations of the SPLP (for each value of the penalty parameter p of the SIMP method), it is necessary to use a large number of vectors (around 15) to obtain $\tilde{\mathbf{u}}$ in such way to achieve a relative magnitude of the residual forces less than 10^{-2}. Naturally, it produces a considerable increasing in the computational effort of the algorithm, and, consequently, its general performance is spoiled. Besides, when $p = 2$, we observed that the number of iterations for which it is necessary to use more vectors to find $\tilde{\mathbf{u}}$ increases considerably. Thus, based on this experience, we propose an alternative scheme: for each value of p, we update the Cholesky factorization of $\mathbf{K}$ in each one of the first 10 iterations of the optimization algorithm when $p = 1$ and $p = 3$, and in the first 50 iterations when $p = 2$. Apart from that, we obtain $\tilde{\mathbf{u}}$ in Eq. (12) through the combined approximations strategy always generating $s = 3$ vectors using Eq. (11), and updating the factorization at each 10 iterations. If the relative magnitude of the residual forces in Eq. (19) is greater than 10^{-2}, we discard $\tilde{\mathbf{u}}$ and we update the Cholesky factorization of $\mathbf{K}$ to evaluate the true vector of nodal displacements $\mathbf{u}$.

Stopping criterion for the SPLP method. In this work, we adopt a mathematically well-founded stopping criterion for the SPLP method. Let

$$\mathcal{L}(\rho,\, \lambda) \,=\, F(\rho) + \lambda V(\rho)$$

be the Lagrangian function associated to the topology optimization problems in Eq. (1) or in Eq. (18), where $F(\rho)$ and $V(\rho)$ denotes, respectively, the objective function and the volume constraint of these problems, and $\lambda \in R$ is the Lagrange multiplier associated to the volume constraint.

Consider the projected gradient $g_P(\rho,\, \lambda)$ of this Lagrangian function onto the set

$$X = \{\rho \in R^n \;\mid\; \rho_{\min} \leq \rho_i \leq 1\,, \qquad i = 1,\, \ldots,\, n\}\,.$$

We consider that the SPLP algorithm found a good approximation for a stationary point for the topology optimization problem in Eq. (1) or in Eq. (18) whenever

$$\|g_P(\rho^{(k)},\, \lambda^{(k)})\|_\infty < 10^{-4}\,,$$

where $\|\,\cdot\,\|_\infty$ denotes the max-norm. In addition to this criterion, we also limit to 200 the number of iterations of the SPLP algorithm for $p = 1$ and $p = 2$, and this limit is increased to 5000 when $p = 3$.

3.2 Results

In this work, we analyse the performance of our alternative scheme (described in Subsection 3.1) based on the combined approximations approach in a benchmark structure: the *cantilever beam*.

The tests were performed on a personal computer with an Intel Core i7-3612QM processor, under the Windows 7 operating system, and the algorithm was implemented in MATLAB. The domain that contains the structure was discretized into four-node rectangular finite elements, using bilinear interpolating functions to approximate the displacements.

Consider the rectangular domain shown in Fig. 1, which has a basis of $0.6\,m$ and a height

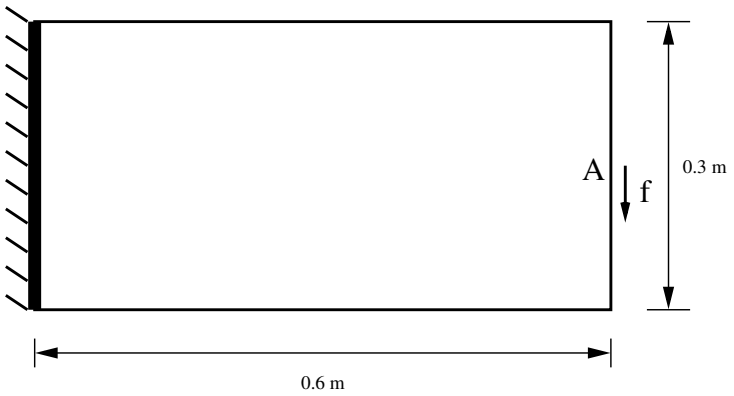

Figure 1: Design domain for the cantilever beam.

of $0.3\,m$. The thickness of the structure is $0.01\,m$. A load of $1\,N$ is applied downwards at the midpoint of the right edge of the domain. The Young's modulus and the Poisson's ratio of the material are set to $10000\,N/m^2$ and 0.3, respectively. The optimal structure must contain no more than $40\,\%$ of the domain's volume.

In order to investigate how the performance of the use of the combined approximations approach within the SPLP method is affected by the number of variables of the topology optimization problem in Eq. (1), we considered the cantilever beam with three different meshes: 60×30 (1800 elements), 120×60 (7200 elements), and 140×70 (9800 elements). The radius adopted for the density filter was set to the length of 2.5, 5.0 and 6.0 elements, respectively.

Table 1: Results for the cantilever beam.

	60×30		120×60		140×70	
	Original	CA	Original	CA	Original	CA
Iterations	476	504	545	549	706	654
Factorizations	476	137	545	130	706	137
Time (s)	230.23	259.21	1663.30	1712.30	3569.70	3514.50
Objective function	95.3175	95.3176	95.9895	95.9895	96.7566	96.7566

The results obtained for each discretization are shown in Table 1. In the columns named *"Original"*, we refer to the solutions found through the updating of Cholesky factorization of $\mathbf{K}$ at each iteration, and in the columns named *"CA"*, we report the solutions obtained when the combined approximations strategy is adopted, following the alternative scheme described in Subsection 3.1. The optimum topologies for the cantilever beam using the different discretizations are presented in Fig. 2, Fig. 3 and Fig. 4.

We can observe from Table 1 that the values of the objective function are very similar for each tested mesh. For the 60×30 and 120×60 meshes, the number of iterations performed by the SPLP algorithm is slightly larger when the CA strategy is adopted, except in the case of 140×70 mesh, for which we note that the SPLP algorithm with the CA approach took fewer iterations and less time.

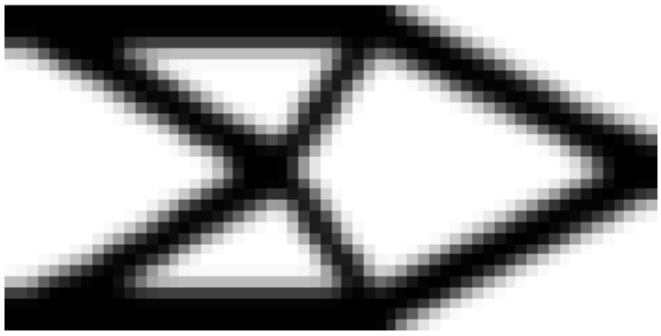

Figure 2: Optimum design for the cantilever beam (60 $\times$ 30 mesh).

Figure 3: Optimum design for the cantilever beam (120 $\times$ 60 mesh).

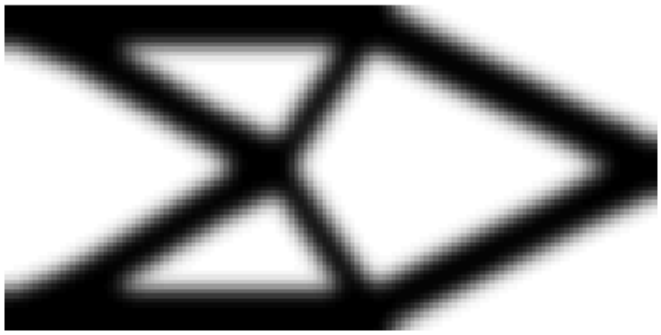

Figure 4: Optimum design for the cantilever beam (140 $\times$ 70 mesh).

When we solve the original topology optimization in Eq. (1), the evaluation of the objective function involves the solution of the large linear system in Eq. (3). On the other hand, when we adopt the CA strategy, we obtain the problem in Eq. (18). In this case, if we always use a fixed number s of vectors in Eq. (11) to obtain $\tilde{\mathbf{u}}$, it is necessary to solve the small linear system in Eq. (17) to compute the objective function in Eq. (18). Having this in mind, when we adopt the CA strategy, we compared the time spent to evaluate the objective function at the iterations where the Cholesky factorization of $\mathbf{K}$ is updated (and, thereby, we solve the linear system in Eq. (3)) with the time spent at iterations when we reuse the factorization (and, therefore, we solve the linear system in Eq. (17)). These data are shown in Table 2, where the columns named *"Update"* are associated to the iterations in which the Cholesky decomposition is updated, whereas the columns denoted by *"Reuse"* concern the iterations in which this factorization is reused.

Analyzing Table 2, we note that the adoption of the CA approach produced a reduction in the time for computing the objective function of the topology optimization problem. With the aim of observing the time spent to evaluate the objective function at each iteration of the SPLP algorithm, a stem plot for each discretization tested for the cantilever beam is shown in Fig. 6, Fig. 7 and Fig. 8. In each one of these plots, the iterations between 1 to 200 are associated to $p = 1$. For $p = 2$, we have the iterations between 201 and 400, and the remaining iterations refer to $p = 3$.

Table 2: Time spent (in seconds) per iteration to evaluate the objective function when CA approach is adopted.

	60×30		120×60		140×70	
	Update	Reuse	Update	Reuse	Update	Reuse
Average time	0.0877	0.0653	0.4654	0.3659	0.6890	0.5500
Maximum time	0.1250	0.0928	0.5869	0.4403	0.8378	0.6829
Minimum time	0.0754	0.0579	0.4247	0.3412	0.6438	0.5290

As we mentioned in Subsection 3.1, we initially allowed a large number of vectors ($s_{\max} = 25$) to obtain the approximated vector of displacements $\tilde{\mathbf{u}}$. In our initial tests, we observed that, especially when $p = 2$ (as we can see in Fig. 5), the algorithm performed several iterations (around 50) in which it spent too much time to evaluate the objective function, because of the large number of vectors used to find $\tilde{\mathbf{u}}$. Based on this fact, we established in our numerical tests that, for $p = 1$ and $p = 3$, the Cholesky factorization of the global stiffness matrix $\mathbf{K}$ is always updated in the first 10 iterations, and, for $p = 2$, the same occurs in the first 50 iterations. Using this criterion, we can observe that, according to the plots in Fig. 6, Fig. 7 and Fig. 8, after these initial iterations for each value of p, the Cholesky factorization of $\mathbf{K}$ was updated a few times (at each 10 iterations), and the number of iterations in which we needed to discard the approximated vector of displacements $\tilde{\mathbf{u}}$ to compute the true vector $\mathbf{u}$ was very small.

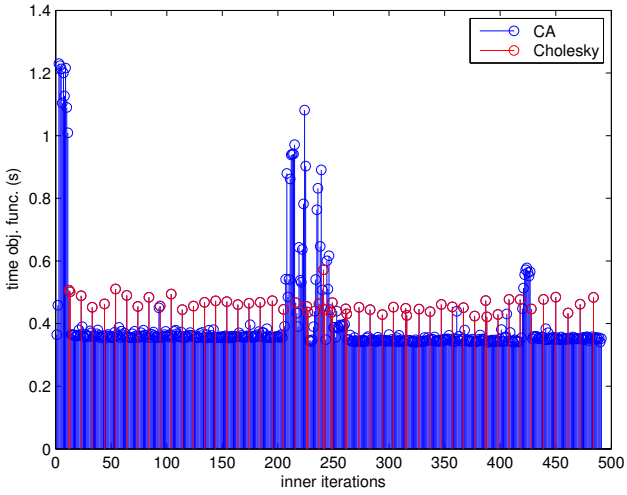

Figure 5: Time spent by the algorithm to evaluate the objective function for the cantilever beam at each iterations when we adopt $s_{\min} = 3$ and $s_{\max} = 25$ (120 $\times$ 60 mesh).

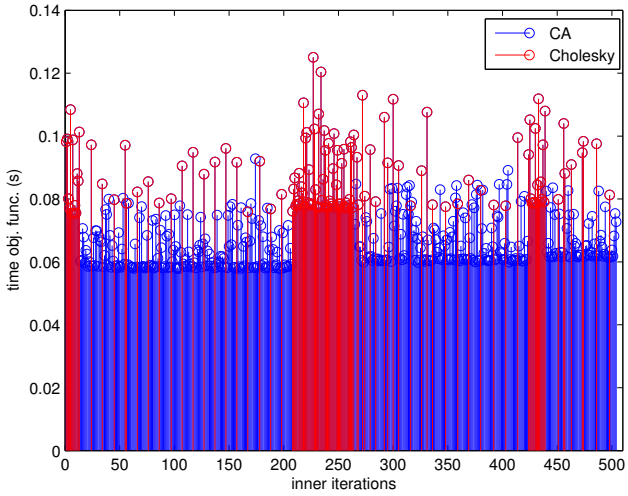

Figure 6: Time spent by the algorithm to evaluate the objective function for the cantilever beam at each iteration (60 $\times$ 30 mesh).

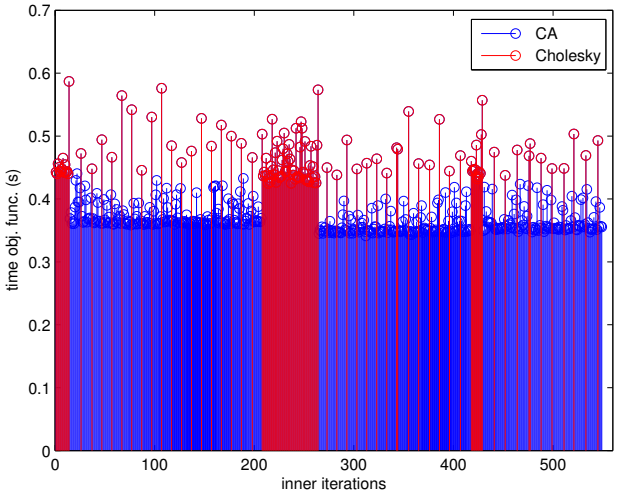

Figure 7: Time spent by the algorithm to evaluate the objective function for the cantilever beam at each iteration (120 $\times$ 60 mesh).

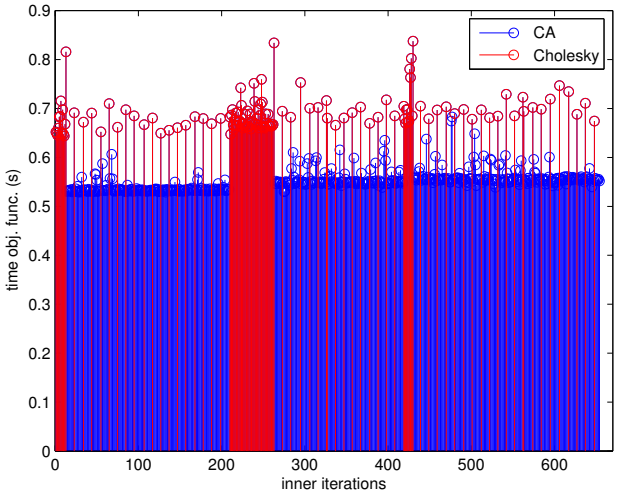

Figure 8: Time spent by the algorithm to evaluate the objective function for the cantilever beam at each iteration (140 $\times$ 70 mesh).

4 CONCLUSIONS

With the goal of reducing the computational effort of evaluating the objective function of topology optimization of structures, we have presented an algorithm that uses the combined approximations approach (inspired by the work of Amir, Bendsøe and Sigmund, 2009) in conjunction with the Sequential Piecewise Linear Programming, presented by Gomes and Senne (2014).

The preliminary numerical tests suggest that the algorithm proposed provides good results, without producing a large variation in the number of iterations performed by the SPLP algorithm, what is promising. From the results summarized in Table 1, we expect that a more significant impact may be obtained for problems with larger dimensions.

As a future work, we can verify the possibility of extending this strategy to topology optimization of structures subjected to large displacements.

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