Calculation of Interaction between Ferromagnetic Solids Simplified by Symmetry Considerations

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Resumo

We discuss the integration of magnetic interaction between spheres and other solids. We show how to use tensors and their eigenvalues in that calculation in order to take advantage of symmetry properties to simplify the expressions. The calculation is gradually restricted to more symmetrical structures, until we reach the case of two homogeneously magnetized spheres arbitrarily positioned and oriented. This work may help future calculations of interaction between more complex objects, such as spheres with inhomogeneous magnetization, ellipsoids and partially paramagnetic solids.

Keywords: ferromagnetism, spheres, solid, magnetization

I. INTRODUCTION

Amikam Aharoni [1] calculated the interaction energy between a ferromagnetic sphere and a paramagnetic one. The ferromagnetic sphere presents magnetization along a fixed direction in its whole volume, while the paramagnetic sphere presents magnetization aligned with the external applied magnetic field in each point of its volume. In order to perform that calculation, he used the propriety that the magnetic field that a uniformly magnetized sphere generates *outside* its volume is equivalent to that of a point dipole in its center [2]. Although not very simple, that calculation is facilitated by the fact that the magnetization is considered to be completely aligned with the applied magnetic field, which means that the core of the calculation is integrating the field generated by a dipole over a spherical volume of arbitrary radius and position.

The field that a ferromagnetic sphere generates in the volume of another ferromagnetic sphere, however, is not aligned (in general) with the magnetization of the latter. Usually, some theorems are used to simplify

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the calculation, but they are not applicable if the system differs from two perfect spheres with continuous, fixed bulk magnetization. Magnetic nanoparticles, for instance, might have *size effects* [3, 4], such as weakening of magnetization towards their borders. In order to calculate the interaction between two solids with those kinds of specific behavior, it is very helpful to be able to calculate the interaction between simpler solids by direct integration over their volume elements. In order to be able to generalize the calculation, we arrive at that same result by direct integration over infinitesimal interacting volume elements, and we show that a tensorial approach might be more useful for some systems.

II. From dipolar field to dipolar interaction

One could argue that, as the magnetic field generated by a ferromagnetic sphere is "dipolar" in the outside region, it is obvious that the interaction energy between two ferromagnetic spheres is also "dipolar", i.e., the interaction energy between two point dipoles. However, this is not that straightforward if we consider basic electrostatic calculations. That conclusion could be taken from Newton's Third Law, $\vec{F}_{AB} = -\vec{F}_{BA}$, where $\vec{F}_{AB}$ is the force in sphere A due to sphere B (and vice-versa for $\vec{F}_{BA}$), or, in terms of potential energy, $\frac{dU(\vec{r}_{AB})}{dx_A} = -\frac{dU(\vec{r}_{BA})}{dx_B}$, where U is the potential energy, $\vec{r}_{AB}$ is the position of the center of sphere A in relation to the center of sphere B (and vice-versa) and x is a Cartesian coordiante. This only restricts U to a certain class of functions.

As the energy interaction between magnetic objects also generates rotation of those objects about axes passing through their own centers, we need to consider the "Rotational Newton's Third Law" [5] as well:

$$\vec{\tau}_{AB} = -\vec{\tau}_{BA}, \tag{1}$$

where τ's are the two spheres' reciprocal torques. The torque generated by a field $\vec{\mathcal{B}}$ on a point dipole m is $\tau = -\vec{m} \times \vec{\mathcal{B}}$, so, Eq. 1 can be integrated as

$$\int_{V_A} \left[\vec{M}_a \times \vec{\mathcal{B}}_{aB}\right] dV_a = -\int_{V_B} \left[\vec{M}_b \times \vec{\mathcal{B}}_{bA}\right] dV_b, \tag{2}$$

where the V's are the volumes of the spheres. It is know that "the average magnetic field due to steady currents outside the sphere is the same as the field they produce at the center" (see *exercise 5.57* of Ref. [6]). Considering that the field generated by each sphere is due to steady currents in their atoms, and that the $\vec{M}$'s are fixed inside each sphere, Eq. 2 becomes

$$\vec{m}_A \times \vec{\mathcal{B}}_{AB} = -\vec{m}_B \times \vec{\mathcal{B}}_{BA}.$$

This equation is a better indication that the spheres interact as dipoles, at least with respect to rotation.

III. Magnetostatic interaction between two spheres

An explicit direct calculation of the interaction between two perfect ferromagnetic spheres might be useful not only to show that the result is dipolar, but also to extend the calculation to other types of magnetization distribution. For instance, there are magnetic nanoparticles that are gradually less ferromagnetic from its center towards the surface along the radial direction, what could be expressed by means of a radial function inside the integrals.

In order to perform the integration between two ferromagnetic spheres, we are going to consider that some sphere A generates a dipolar magnetic field that interacts with infinitesimal magnetic dipole moments that occupy the whole volume of a sphere B. That is to say that we are going to integrate, over the volume of sphere B, the magnetic interaction energy between two point magnetic dipoles:

$$U_{mag(ij)} = \frac{\mu}{4\pi}\left(\frac{\vec{m}_i \cdot \vec{m}_j}{r_{ji}^3} - 3\frac{(\vec{m}_i \cdot \vec{r}_{ji})(\vec{m}_j \cdot \vec{r}_{ji})}{r_{ji}^5}\right),$$

where μ is the mean magnetic permeability, $\vec{m}_i$ and $\vec{m}_j$ are the magnetic dipoles of two interacting particles i and j, $\vec{r}_{ji}$ is the vector that describes the position of particle j in relation to particle i.

So, the energy between two infinitesimal dipoles, one in sphere A and the other in sphere B, is written as

$$dU_{ab} = \frac{\mu}{4\pi}\left[\frac{d\vec{m}_a \cdot d\vec{m}_b}{r_{bA}^3} - \frac{3(d\vec{m}_a \cdot \vec{r_{bA}})(d\vec{m}_b \cdot \vec{r_{bA}})}{r_{bA}^5}\right], \tag{3}$$

The magnetic dipolar field, generated by the first sphere of total magnetic dipole $\vec{m}_A$ must be integrated over the volume of the second one, of constant volume magnetization $\vec{M}_B$ (magnetic dipole per volume)

$$U_{AB} = \frac{\mu}{4\pi} m_A M_A \int_{V_b} \frac{1}{r_{bA}^3}\left\{\hat{m}_A \cdot \hat{m}_B - 3(\hat{m}_A \cdot \hat{r}_{bA})(\hat{m}_B \cdot \hat{r}_{bA})\right\} dV_B, \tag{4}$$

where m_A is the total dipole of sphere A; M_B is the magnetization of sphere B and is written as a function of m_B and V_B as $M_B = m_B/V_B$, (or, in the general case of magnetization that may vary over volume, $M_B = \frac{dm_b}{dV_b} \Rightarrow dm_B = M_B dV_B$); $\vec{r}_{bA}$ is the position of a volume element of sphere B in relation to the center of sphere A.

Now we define the direction of the vectors $\vec{m}_A$, $\vec{m}_B$ and $\vec{r}$ in terms of spherical coordinate angles:

$$\hat{m}_A = \sin\theta_{mA}\cos\phi_{mA}\hat{x} + \sin\theta_{mA}\sin\phi_{mA}\hat{y} + \cos\theta_{mA}\hat{z},$$
$$\hat{m}_B = \sin\theta_{mB}\cos\phi_{mB}\hat{x} + \sin\theta_{mB}\cos\phi_{mB}\hat{y} + \cos\theta_{mB}\hat{z},$$
$$\hat{r}_{bA} = \sin\theta_{bA}\cos\phi_{bA}\hat{x} + \sin\theta_{bA}\cos\phi_{bA}\hat{y} + \cos\theta_{bA}\hat{z},$$

from which it is easy to see that

$$\hat{m}_A \cdot \hat{m}_B = \cos\theta_{mA}\cos\theta_{mB} + \sin\theta_{mA}\sin\theta_{mB}\cos(\phi_{mA} - \phi_{mB}),$$
$$\hat{m}_{mA} \cdot \hat{r}_{bA} = \cos\theta_{mA}\cos\theta_{bA} + \sin\theta_{mA}\sin\theta_{bA}\cos(\phi_{mA} - \phi_{bA}),$$
$$\hat{m}_{mB} \cdot \hat{r}_{bA} = \cos\theta_{mB}\cos\theta_{bA} + \sin\theta_{mB}\sin\theta_{bA}\cos(\phi_{mB} - \phi_{bA}).$$

As the integration will be performed over the volume of sphere B, it is necessary to write $\vec{r}_{bA}$, the position of a volume element of sphere B in relation to the center of sphere A (the origin of our coordinate system), as a function of $\vec{r}_{bB}$, the position of a volume element of sphere B in relation to the center of sphere B itself,

$$\vec{r}_{bA} = r_{bB}\sin\theta_{bB}\cos\phi_{bB}\hat{x} + r_{bB}\sin\theta_{bB}\sin\phi_{bB}\hat{y} + (r_{BA} + r_{bB}\cos\theta_{bB})\hat{z}.$$

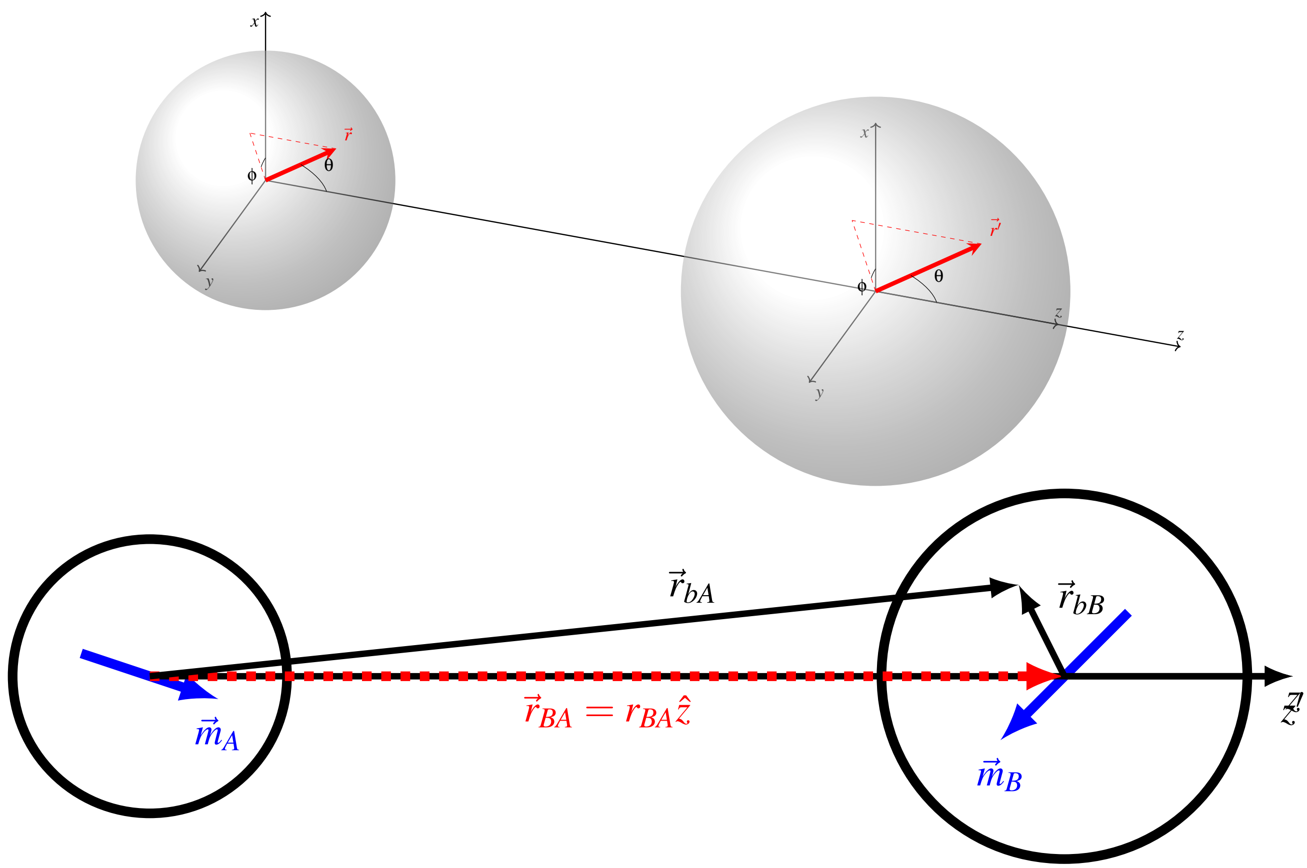

Figura 1: *Spherical coordinate systems for integration over the volume of two interacting spheres, with a cross section showing the position of a volume element of sphere B in relation to the center of both spheres A and B. In spherical coordinates,* $\vec{r}_{bA} \equiv (r_{bA}, \theta_{bA}, \phi_{bA})$ *and* $\vec{r}_{bB} \equiv (r_{bB}, \theta_{bB}, \phi_{bB})$.

From the identity $r^2 = \vec{r}\cdot\vec{r}$ applied to $\vec{r}_{bA}$, we get

$$r_{bA} = (r_{bB}^2 + r_{BA}^2 + 2r_{bB}r_{BA}\cos\theta_{bB})^{1/2}. \tag{5}$$

Equations 4, 5 and 5 yield

$$U_{AB} = \frac{\mu}{4\pi} m_A M_B \int_{V_B} \left[\frac{A - 3B(\phi_{bA})C(\phi_{bA})}{(r'^2 + R'^2 + 2r'R'\cos\theta')^{3/2}}\right] dV_B,$$

where

$$\begin{aligned}
A &= \cos\theta_{mA}\cos\theta_{mB} + \sin\theta_{mA}\sin\theta_{mB}\cos(\phi_{mA} - \phi_{mB}),\\
B(\phi_{bA}) &= \cos\theta_{mA}\cos\theta_{bA} + \sin\theta_{mA}\sin\theta_{bA}\cos(\phi_{mA} - \phi_{bA}),\\
C(\phi_{bA}) &= \cos\theta_{mB}\cos\theta_{bA} + \sin\theta_{mB}\sin\theta_{bA}\cos(\phi_{mB} - \phi_{bA}),
\end{aligned}$$

where the dependence on θ_{bA} is highlighted because it is the only angle above that will vary in the integration. By performing the multiplication $B(\phi_{bA})C(\phi_{bA})$, and considering that, by symmetry, the integration of terms with single $\cos(\phi_{mB} - \phi_{bA})$ or $\cos(\phi_{mA} - \phi_{bA})$ (but not that with the two of them multiplied) equals zero, we get

$$\begin{aligned}
U_{AB} =& \frac{\mu}{4\pi} m_A M_B \left\{ \int_{V_B} \frac{A}{(r_{bB}^2 + r_{BA}^2 + 2r_{bB}r_{BA}\cos\theta_{bB})^{3/2}} dV_b \right.\\
&- 3\left[\cos\theta_{mA}\cos\theta_{mB}\int_{V_B}\frac{\cos^2\theta_{bA}}{(r_{bB}^2 + r_{BA}^2 + 2r_{bB}r_{BA}\cos\theta_{bB})^{3/2}} dV_b\right.\\
&\left.\left. +\ \sin\theta_{mA}\sin\theta_{mB}\int_{V_B}\frac{\sin^2\theta_{bA}\cos(\phi_a - \phi_{bA})\cos(\phi_b - \phi_{bA})}{(r_{bB}^2 + r_{BA}^2 + 2r_{bB}r_{BA}\cos\theta_{bB})^{3/2}} dV_b\right]\right\},
\end{aligned} \tag{6}$$

We must change the numerators' dependence from r_{bA}, θ_{bA}, ϕ_{bA} to r_{bB}, θ_{bB}, ϕ_{bB} in order to be able to integrate the dipolar field from $\vec{m}_A$ over V_B. A straightforward vector addition yields, for the z components (remember that $r\hat{_{AB}}$ is along the z-axis), $r_{bA}\cos\theta_{bA} = r_{AB} + r_{bA}\cos\theta_{bB}$ that squared becomes

$$(r_{bA}\cos\theta_{bA})^2 = r_{AB}^2 + 2r_{AB}r_{bB}\cos\theta_{bB} + r_{bB}^2\cos^2\theta_{bB}. \tag{7}$$

Note also that the projections of $\vec{r}_{bB}$ and $\vec{r}_{bA}$ on the plane xy must be equal to each other because our choice of axes implies that Cartesian positions centered on the two sphere only differ in the z-component but not in the x and y components. So,

$$r_{bB}\sin\theta_{bB} = r_{bA}\sin\theta_{bA}, \tag{8}$$

and $\phi_{bA} = \phi_{bB}$, such that

$$r_{bA}^2\sin^2\theta_{bA}\cos(\phi_{mA} - \phi_{bA})\cos(\phi_{mB} - \phi_{bA}) = r_{bB}^2\sin^2\theta_{bB}\cos(\phi_{mA} - \phi_{bB})\cos(\phi_{mB} - \phi_{bB}). \tag{9}$$

Now, eq. 7 and eq. 9 may be used in eq. 6 to change the origin of its position coordinates from the center of sphere A to the center of sphere B.

$$\begin{aligned} U_{AB} = \frac{\mu}{4\pi} m_A M_B \Bigg\{ \int_{V_B} \frac{A}{(r_{bB}^2 + r_{BA}^2 + 2 r_{bB} r_{BA} \cos\theta_{bB})^{3/2}} dV_b \\ -3 \left[\cos\theta_{mA} \cos\theta_{mB} \int_{V_B} \frac{r_{AB}^2 + 2 r_{AB} r_{bB} \cos\theta_{bB} + r_{bB}^2 \cos^2\theta_{bB}}{((r_{bB}^2 + r_{BA}^2 + 2 r_{bB} r_{BA} \cos\theta_{bB})^{5/2}} dV_b \right. \\ \left. + \sin\theta_{mA} \sin\theta_{mB} \int_{V_b} \frac{F(r_{bB}, \theta_{bB}, \phi_{bB} | \phi_{mA}, \phi_{mB})}{(r_{bB}^2 + r_{BA}^2 + 2 r_{bB} r_{BA} \cos\theta_{bB})^{5/2}} dV_b \right] \Bigg\}, \end{aligned} \tag{10}$$

where the function F is

$$\begin{aligned} F(r_{bB}, \theta_{bB}, \phi_{bB} | \phi_{mA}, \phi_{mB}) = \sin^2\theta_{bB} \cos(\phi_{mA} - \phi_{bB}) \cos(\phi_{mB} - \phi_{bB}) \\ = \sin^2\theta_{bB} \left[(\cos\phi_{mA} \cos\phi_{mB}) \cos^2\phi_{bB} + (\sin(\phi_{mA} + \phi_{mB})) \sin\phi_{bB} \cos\phi_{bB} + (\sin\phi_{mA} \sin\phi_{mB}) \sin^2\phi_{bB} \right] \end{aligned}$$

Two exponents in the denominators of eq. 10 changed from 3/2 to 5/2 because a factor r_{bA}^2 have been added in both numerators and denominators in order to use eq. 7 and eq. 9, and the one in the denominator has been replaced again with eq. 5. Now, Eq. 10 does not depend on any variable centered on sphere A, but only on quantities that will remain constant during the integration (r_{AB}, θ_{AB}, θ_{mB}, ϕ_{mA}, ϕ_{mB}) and on variables centered on sphere B (r_{bB}, θ_{bB}, ϕ_{bB}). So, as we do not have variables with indices "bA" and "bB" to distinguish anymore, we can omit the index "bB" from now on and define $dV_b = r^2 \sin\theta dr d\theta d\phi$. By doing that and changing the variables as

$$[\xi = \cos\theta] \Rightarrow [d\xi = -\sin\theta d\theta] \Rightarrow \left[\int_0^\pi \sin\theta d\theta \to -\int_1^{-1} d\xi \to \int_{-1}^1 d\xi \right], \tag{11}$$

we arrive at

$$\begin{aligned} U_{AB} = \frac{\mu m_A M_B}{4\pi} \Bigg\{ (\hat{m}_A \hat{m}_B) 2\pi \int_0^{R_B} \int_{-1}^1 \frac{r^2}{(r^2 + r_{BA}^2 + 2 r r_{BA} \xi)^{3/2}} d\xi dr \\ -3(\cos\theta_{mA} \cos\theta_{mB}) 2\pi \int_0^{R_B} \int_{-1}^1 \frac{(r_{BA}^2 + 2 r r_{BA} \xi + r^2 \xi^2) r^2}{(r^2 + r_{BA}^2 + 2 r r_{BA} \xi)^{5/2}} d\xi dr, \\ -3(\sin\theta_{mA} \sin\theta_{mB}) \int_0^{R_B} \int_{-1}^1 \frac{r^4 (1 - \xi^2)}{(r^2 + r_{AB}^2 + 2 r r_{AB} \xi)^{5/2}} d\xi dr \\ \times \int_0^{2\pi} \left[\cos\phi_{mA} \cos\phi_{mB} \cos^2\phi + \sin(\phi_{mA} + \phi_{mB}) \cos\phi \sin\phi + \sin\phi_{mA} \sin\phi_{mB} \sin^2\phi \right] d\phi \Bigg\} \end{aligned} \tag{12}$$

where R_A and R_B are the radii of the spheres A and B.

After some manipulation (and using integral tables) one can simply that expression to

$$\begin{aligned} U_{AB} = \frac{\mu}{4\pi} m_A \times \left(\frac{4\pi}{3} R_B^3 M_B \right) \times \frac{1}{r_{AB}^3} \times \{ -2\cos\theta_{mA} \cos\theta_{mB} + \sin\theta_{mA} \sin\theta_{mB} \cos(\phi_{mA} - \phi_{mB}) \} \\ = \frac{\mu}{4\pi} m_A \times \left(\frac{4\pi}{3} R_B^3 M_B \right) \times \frac{1}{r_{AB}^3} \times \{ \cos\theta_{mA} \cos\theta_{mB} + \sin\theta_{mA} \sin\theta_{mB} \cos(\phi_{mA} - \phi_{mB}) - 3\cos\theta_{mA} \cos\theta_{mB} \} \end{aligned}$$

and, finally, recalling that $\hat{m}_A \cdot \hat{m}_B = \cos\theta_{mA}\cos\theta_{mB} + \sin\theta_{mA}\sin\theta_{mB}\cos(\phi_{mA} - \phi_{mB})$ (eqs. 5), $cos\theta_{mA}\cos\theta_{mB} = (\hat{m}_A \cdot \hat{z})(\hat{m}_B \cdot \hat{z}) = (\hat{m}_A \cdot \hat{r}_{BA})(\hat{m}_B \cdot \hat{r}_{BA})$, the equation above becomes

$$U_{AB} = \frac{\mu}{4\pi} \times \frac{\vec{m}_A \vec{m}_B - 3(\vec{m}_A \cdot \hat{r}_{BA})(\vec{m}_B \cdot \hat{r_{BA}})}{r_{BA}^3}$$

The magnetic interaction between two **perfectly spherical particles** with **completely homogeneous magnetization** is equivalent to that of two point dipoles located at the center of those particles. This expression also ignores particles' **crystal structure**, which means that the value of the interaction should be kept constant for surface-surface distances **smaller than lattice constant**; fortunately, this interaction does not increase that much for small distances and the inclusion of the effect of crystal structure may be superfluous. **Magnetic domains**, **partial paramagnetism** and **deviation from sphericity** might also get the real interaction away from the dipolar model.

IV. The tensorial approach

Due to the availability of many *computer algebra systems* (CAS) that perform matrix calculations, a tensorial approach is useful to check and generalize the previous results. To do that, we write the second term of eq. 3 in dyadic form $\vec{r}_{ba}\vec{r}_{ba}$ such that:

$$(d\vec{m}_a \cdot \vec{r}_{ba})(d\vec{m}_b \cdot \vec{r}_{ba}) = d\vec{m}_a \cdot \vec{r}_{ba}\vec{r}_{ba} \cdot d\vec{m}_b,$$

where

$$\vec{r}_{ba}\vec{r}_{ba} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} x & y & z \end{bmatrix} = \begin{bmatrix} x^2 & xy & xz \\ yx & y^2 & yz \\ zx & zy & z^2 \end{bmatrix},$$

which is a symmetric second order tensor, so that Eq. 3 can be rewritten as

$$dU_{ab} = \frac{\mu}{4\pi}\left[\frac{d\vec{m}_a \cdot d\vec{m}_b}{r_{ba}^3} - \frac{3(d\vec{m}_a \cdot \vec{r}_{ba}\vec{r}_{ba} \cdot d\vec{m}_b)}{r_{ba}^5}\right], \tag{13}$$

and finally

$$dU_{ab} = \frac{\mu}{4\pi}\left\{d\vec{m}_a \cdot \left[\frac{r_{ba}^2 \mathbb{1} - 3\vec{r}_{ba}\vec{r}_{ba}}{r_{ba}^5}\right] \cdot d\vec{m}_b\right\}, \tag{14}$$

where $\mathbb{1}$ is identity matrix.

Eq. 14 may be also written as

$$dU_{ab} = d\vec{m}_a \cdot \mathbb{E} \cdot d\vec{m}_b,$$

where $\mathbb{E}$ could be described as **the energy operator in the basis of magnetic moment** by those who are familiar with Quantum Mechanics calculations. For this very reason, the operator $\mathbb{E}$ is independent of the magnetic moments. It may be written as

$$\mathbb{E} = \frac{\mu}{4\pi}\left[\frac{\mathbb{K}}{r_{ba}^5}\right],$$

where $\mathbb{K} = r_{ba}^2 \mathbb{1} - 3\vec{r}_{ba}\vec{r}_{ba}$.

The eigenvalues of the tensor $\mathbb{K}$ are $\lambda_1 = x^2+y^2+z^2$, with multiplicity **two**, and $\lambda_2 = -2(x^2+y^2+z^2)$, with multiplicity **one**, such that $\mathbb{K}$ can be decomposed by means of the so-called "eigen decomposition" as

$$\mathbb{K} = \mathbb{P}\mathbb{D}\mathbb{P}^{-1} \tag{15}$$

where

$$\mathbb{D} = \begin{bmatrix} x^2+y^2+z^2 & 0 & 0 \\ 0 & x^2+y^2+z^2 & 0 \\ 0 & 0 & -2(x^2+y^2+z^2) \end{bmatrix},$$

the matrix P is formed with the eigenvectors of $\mathbb{K}$: $\vec{v}_{\lambda_{1,1}} = -\frac{\cos\theta}{\sin\theta\sin\phi}\hat{y} + \hat{z}$, $\vec{v}_{\lambda_{1,2}} = \hat{x} - \frac{\cos\phi}{\sin\phi}\hat{y}$ and $\vec{v}_{\lambda_2} = \frac{\sin\theta\cos\phi}{\cos\theta}\hat{x} + \frac{\sin\theta\sin\phi}{\cos\theta}\hat{y} + \hat{z}$, that is,

$$\mathbb{P} = \begin{bmatrix} 0 & 1 & \frac{\sin\theta\cos\phi}{\cos\theta} \\ -\frac{\cos\theta}{\sin\theta\sin\phi} & -\frac{\cos\phi}{\sin\phi} & \frac{\sin\theta\sin\phi}{\cos\theta} \\ 1 & 0 & 1 \end{bmatrix},$$

and the inverse matrix $\mathbb{P}^{-1}$ is

$$\mathbb{P}^{-1} = \begin{bmatrix} -\sin\theta\cos\theta\cos\phi & -\sin\theta\cos\theta\sin\phi & \sin^2\theta \\ \sin^2\theta\sin^2\phi + \cos^2\theta & -\sin^2\theta\sin\phi\cos\phi & -\sin\theta\cos\theta\cos\phi \\ \sin\theta\cos\theta\cos\phi & \sin\theta\cos\theta\sin\phi & \cos^2\theta \end{bmatrix}.$$

Eqs. 15 and 14 implies that

$$\begin{aligned} dU_{ab} &= \frac{\mu}{4\pi}\left\{d\vec{m}_a \cdot \left[\mathbb{P}\mathbb{D}\mathbb{P}^{-1}\right] \cdot d\vec{m}_b\right\} \\ &= \frac{\mu}{4\pi}\left\{\left[d\vec{m}_a \cdot \mathbb{P}\right]\left[\mathbb{D}\right]\left[\mathbb{P}^{-1} \cdot d\vec{m}_b\right]\right\}, \end{aligned}$$

where $d\vec{m}_a \cdot \mathbb{P}$ is written in matrix form as a row vector,

$$d\vec{m}_a \cdot \mathbb{P} = \begin{bmatrix} a & b & c \end{bmatrix},$$

where

$$\begin{aligned} a &= -\frac{\cos\theta}{\sin\theta\sin\phi}\, dm_{ay} + dm_{az}, \\ b &= dm_{ax} - \frac{\cos\phi}{\sin\phi}\, dm_{ay}, \\ c &= \frac{\sin\theta\cos\phi}{\cos\theta}\, dm_{ax} + \frac{\sin\theta\sin\phi}{\cos\theta}\, dm_{ay} + dm_{az}, \end{aligned}$$

and $\mathbb{P}^{-1} \cdot d\vec{m}_b$ as a column vector,

$$\mathbb{P}^{-1} \cdot d\vec{m}_b = \begin{bmatrix} d \\ e \\ f \end{bmatrix}.$$

where

$$
\begin{aligned}
d =& -\sin\theta\cos\theta\cos\phi\ dm_{bx} - \sin\theta\cos\theta\sin\phi\ dm_{by} + \sin^2\theta\ dm_{bz},\\
e =& (\sin^2\theta\sin^2\phi + \cos^2\theta)\ dm_{bx} - \sin^2\theta\sin\phi\cos\phi\ dm_{by} - \sin\theta\cos\theta\cos\phi\ dm_{bz},\\
f =& \sin\theta\cos\theta\cos\phi\ dm_{bx} + \sin\theta\cos\theta\sin\phi\ dm_{by} + \cos^2\theta\ dm_{bz}
\end{aligned}
$$

Since $\mathbb{D}$ is a diagonal matrix, the components of $d\vec{m}_a\cdot\mathbb{P}$ and $\mathbb{P}^{-1}\cdot d\vec{m}_b$ will be multiplied separately along each axis, without "cross-multiplication". After some algebraic manipulation,

$$
\begin{aligned}
&[d\vec{m}_a\cdot\mathbb{P}][\mathbb{D}][\mathbb{P}^{-1}\cdot d\vec{m}_b]\\
&= r^2\left[(1-3\sin^2\theta\cos^2\theta)\ dm_{ax}\ dm_{bx} + (\sin\phi\cos\phi(\sin^2\theta+2\sin\theta\cos\theta))\ dm_{ax}\ dm_{by} - 3\ \sin\theta\cos\theta\cos\phi\ dm_{ax}\ dm_{bz}\right.\\
&-3\ \sin^2\theta\sin\phi\cos\phi\ dm_{ay}\ dm_{bx} + (1-3\sin^2\theta\sin^2\theta)\ dm_{ay}\ dm_{by} - 2\ \sin\theta\cos\theta\sin\phi\ dm_{ay}\ dm_{bz}\\
&\left.-3\ \sin\theta\cos\theta\cos\phi\ dm_{az}\ dm_{bx} - 3\ \sin\theta\cos\theta\sin\phi\ dm_{az}\ dm_{by} + (1-3\cos^2\theta)\ dm_{az}\ dm_{bz}\right]
\end{aligned}
\tag{16}
$$

The equation above is quite **general**. It is valid for all sorts of forms. However, r, θ and ϕ are spherical coordinates that position $d\vec{m}_b$ in relation to $d\vec{m}_a$, such that, for a non spherical volume, the integration limits of those variable are interdependent. However, for instance, for ellipsoidal volumes, a reparametrization of the coordinate system may give similar results as the ones obtained by us in the present paper.

V. Form analysis

In Eq. 16, r, θ and ϕ describe the spherical coordinates of the positions of all volume elements of body B in relation to all volume elements of body A. That expression is supposed to be integrated over the volumes of A and B, so one would normally have to change its dependence from $\{r,\theta,\phi\}$ to $\{r_a,\theta_a,\phi_a\},\{r_b,\theta_b,\phi_b\}$. However, even before performing that transformation, it is possible to make some progress in the analysis of bodies with some symmetry properties.

Bodies with rotational symmetry along z:

Let us restrict ourselves to the interaction between bodies whose volumes share a symmetry axis in z. That integration will cancel out all those factors whose dependence in ϕ are $\sin\phi$, $\cos\phi$ or $\sin\phi\cos\phi$, provided that, for each $\{r,\theta\}$ pair, ϕ assume values that are regularly distributed along the cycle $\{0,2\pi\}$. That condition is fulfilled if the bodies A and B share an axis of rotational symmetry along the z axis, as illustrated in Fig. 2, because, in that situation, the integration over ϕ is cancelled out for any pair r,θ.

For systems with that kind of symmetry,

$$
\begin{aligned}
&[d\vec{m}_a\cdot\mathbb{P}][\mathbb{D}][\mathbb{P}^{-1}\cdot d\vec{m}_b]^{(\text{not canceled by } \int d\phi)} = (x^2+y^2+z^2)\\
&\times\left[dm_{ax}dm_{bx}(1-3\sin^2\theta\cos^2\phi) + dm_{ay}dm_{by}(1-3\sin^2\theta\sin^2\phi) + dm_{az}dm_{bz}(1-3\cos^2\phi)\right]\\
&= r^2\left\{d\vec{m}_I d\vec{m}_J - 3\left[(\sin^2\theta\cos^2\phi)dm_{ax}dm_{bx} + (\sin^2\theta\sin^2\phi)dm_{ay}dm_{by} + (\cos^2\theta)dm_{az}dm_{bz}\right]\right\}
\end{aligned}
$$

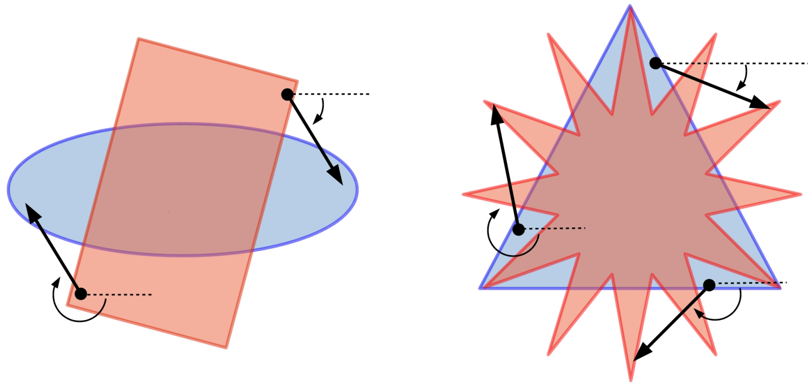

Figura 2: *Transversal-to-z sections of two pairs of bodies which share rotational symmetry axis along z. For each relative position vector, it is possible to find other vectors of same magnitude and orientation changed by displacements of π/n in the relative azimuthal angle, with $n = 2$ and $n = 3$ in the examples.*

such that Eq. 14 becomes, remembering also that $r \equiv r_{ba}$, $\theta \equiv \theta_{ba}$ and $\phi \equiv \phi_{ba}$,

$$dU_{ab} = \frac{\mu}{4\pi}\frac{1}{r_{ba}^3}\left\{d\vec{m}_a \cdot d\vec{m}_b - 3\left[(\sin^2\theta_{ba}\cos^2\phi_{ba})dm_{ax}dm_{bx} \right.\right. \\ \left.\left. +(\sin^2\theta_{ba}\sin^2\phi_{ba})dm_{ay}dm_{by} + (\cos^2\theta_{ba})dm_{az}dm_{bz}\right]\right\}. \quad (17)$$

Now, considering constant magnetization inside each body, we can put

$$d\vec{m}_a = \vec{M}_a dV_a \qquad \text{and} \qquad d\vec{m}_b = \vec{M}_b dV_b$$

in Eq. 17, resulting in

$$dU_{ab} = \frac{\mu}{4\pi}\frac{1}{r_{ba}^3}\left\{\vec{M}_a \cdot \vec{M}_b - 3\left[(\sin^2\theta_{ba}\cos^2\phi_{ba})M_{ax}M_{bx} \right.\right. \\ \left.\left. +(\sin^2\theta_{ba}\sin^2\phi_{ba})M_{ay}M_{by} + (\cos^2\theta_{ba})M_{az}M_{bz}\right]\right\} dV_a dV_b. \quad (18)$$

The only assumption we have made about volumes V_a and V_b so far is that both of them have a rotational symmetry axis along the z-axis, where the center-to-center distance between solids A and B extends along. So, in such a configuration, Eq. 18 is valid for ferromagnetic ellipsoids, cubes, cylinders, tubes (even with varying cross-section), etc.

A point dipole (or a sphere) and a body with rotational symmetry:

Let us replace the body A with a point dipole m_A, so that Eq. 18 becomes

$$dU_{Ab} = \frac{\mu}{4\pi}\frac{1}{r_{bA}^3}\left\{\vec{m}_A \cdot \vec{M}_b - 3\left[(\sin^2\theta_{bA}\cos^2\phi_{bA})m_{Ax}M_{bx} \right.\right. \\ \left.\left. +(\sin^2\theta_{bA}\sin^2\phi_{bA})m_{Ay}M_{by} + (\cos^2\theta_{bA})m_{Az}M_{bz}\right]\right\} dV_b. \quad (19)$$

As already shown, the magnetic field of a sphere with fixed volume magnetization is dipolar as experienced from outside its volume. Thus, Eq. 19 is also applicable to the interaction between a sphere and a ferromagnetic body with a rotational symmetry axis along the distance between their centers.

A point dipole (or a sphere) and a body with circular transversal section:

Now, consider that B is a body like a tube, a cylinder or an ellipsoid, i.e., with circular cross sections (either disks or rings). For such a system, it is convenient to write the volume element in spherical or cylindrical coordinates, but we shall be careful because the conventional volume element forms (e.g. $r^2 \sin\theta d\theta d\phi dr$) are usually expressed in relation to the center of the solid being integrated. Let us again distinguish variables expressed in relation to the center of solid B from the ones expressed in relation to the center of solid A:

$$dU_{Ab} = \frac{\mu}{4\pi}\int\int\int \frac{1}{r_{bA}^3}\Big\{\vec{m}_A \cdot \vec{M}_J - 3\big[(\sin^2\theta_{bA}\cos^2\phi_{bA})m_{Ax}M_{bx} + (\sin^2\theta_{bA}\sin^2\phi_{bA})m_{Ay}M_{by} + (\cos^2\theta_{bA})m_{Az}M_{bz}\big]\Big\} r_{bB}^2 \sin\theta_{bB} d\phi_{bB} dr_{bB} d\theta_{bB}.$$

As the angles ϕ refer to the angular position of projections in the xy-plane, we can define that $\phi_{aA} = \phi_{bB}$ without loss of generality (that is not true for θ). As A is a point dipole at the origin (i.e., $\vec{r}_A = (0,0,0)$), and because of the circular transversal sections of B, the integrations in ϕ are readily done from 0 to 2π:

$$dU_{Ab} = \frac{\mu}{4}\int_0^{R_B}\int_0^{\pi} \frac{1}{r_{bA}^3}\Big\{2(\vec{m}_A \cdot \vec{M}_J) - 3\big[\sin^2\theta_{bA} m_{Ax}M_{bx} + \sin^2\theta_{bA} m_{Ay}M_{by} + 2\cos^2\theta_{bA} m_{Az}M_{bz}\big]\Big\} r_{bB}^2 \sin\theta_{bB} dr_{bB} d\theta_{bB}.$$

To follow this calculation, one must now define the integration limits, but the upper integration limit of r_{bB} depends on θ_{bB}, with specific expressions for each solid. For an ellipsoid, it might be convenient to stand with spherical coordinates, but for cylindrical tubes, one may choose to shift to cylindrical coordinates.

A point dipole and a sphere (or two spheres)

Integrating the interaction energy between two spheres is equivalent to considering one of them a point dipole and integrating its filed over the volume of the other. Let us consider sphere A as a point dipole and integrate its field over the volume of sphere B [1]. Eq. 19 to this system.

When using spherical coordinates, spheres are special because the integration limits of the coordinates are independent of each other:

$$dU_{Ab} = \frac{\mu}{4\pi}\int_0^{R_B}\int_0^{\pi}\int_0^{2\pi} \frac{1}{r_{bA}^3}\Big\{\vec{m}_A \cdot \vec{M}_J - 3\big[(\sin^2\theta_{bA}\cos^2\phi_{bA})m_{Ax}M_{bx} + (\sin^2\theta_{bA}\sin^2\phi_{bA})m_{Ay}M_{by} + (\cos^2\theta_{bA})m_{Az}M_{bz}\big]\Big\} r_{bB}^2 \sin\theta_{bB} d\phi_{bB} d\theta_{bB} dr_{bB}.$$

[1] Alternatively, we could do the opposite: consider sphere B as a point dipole and integrate its field over the volume of sphere A. Newton's Third Law guarantees that we would have the same result.

Recalling eq. 5,

$$dU_{Ab} = \frac{\mu}{4\pi}\int_0^{R_B}\int_0^{\pi}\int_0^{2\pi}\frac{1}{(r_{bB}^2+r_{BA}^2+2r_{bB}r_{BA}\cos\theta_{bB})^{3/2}}$$
$$\Big\{\vec{m}_A\cdot\vec{M}_J-3\left[(\sin^2\theta_{bA}\cos^2\phi_{bA})m_{Ax}M_{bx}\right.$$
$$\left.+(\sin^2\theta_{bA}\sin^2\phi_{bA})m_{Ay}M_{by}+(\cos^2\theta_{bA})m_{Az}M_{bz}\right]\Big\}\, r_{bB}^2\sin\theta_{bB}d\phi_{bB}d\theta_{bB}dr_{bB}.$$

Using $\sin\theta_{bA} = r_{bB}\sin\theta_{bB}/r_{bA}$ (from eq. 8) and $\phi_{bA} = \phi_{bB}$, and then using eq. 7 to remedy the last Ji dependence,

$$\begin{aligned} dU_{Ab} = &\frac{\mu}{4\pi}\int\frac{[\vec{m}_A\cdot\vec{M}_b]r_{bB}^2\sin\theta_{bB}}{(r_{bB}^2+r_{BA}^2+2r_{bB}r_{BA}\cos\theta_{bB})^{3/2}}dV \\ &-\frac{3\mu}{4\pi}\int\frac{[m_{Ax}M_{bx}](r_{bB}^4\sin^3\theta_{bB}\cos^2\phi_{bB})}{(r_{bB}^2+r_{BA}^2+2r_{bB}r_{BA}\cos\theta_{bB})^{5/2}}dV \\ &-\frac{3\mu}{4\pi}\int\frac{[m_{Ay}M_{by}](r_{bB}^4\sin^3\theta_{bB}\sin^2\phi_{bB})}{(r_{bB}^2+r_{BA}^2+2r_{bB}r_{BA}\cos\theta_{bB})^{5/2}}dV \\ &-\frac{3\mu}{4\pi}\int[m_{Az}M_{bz}]r_{bA}^2\frac{(r_{BA}^2\sin\theta_{bB}+2r_{BA}r_{bB}\sin\theta_{bB}\cos\theta_{bB}+r_{bB}^2\sin\theta_{bB}\cos^2\theta_{bB})}{(r_{bB}^2+r_{BA}^2+2r_{bB}r_{BA}\cos\theta_{bB})^{5/2}}dV \end{aligned}$$

where $\int \equiv \int_0^{R_B}\int_0^{\pi}\int_0^{2\pi}$ and $dV \equiv d\phi_{bB}d\theta_{bB}dr_{bB}$.

Considering that the direction of the magnetization is fixed inside each sphere, and performing the integrations in ϕ_{bB} (which are very simple), it is trivial to arrive at

$$\begin{aligned} dU_{iJ} = &\frac{\mu}{4}[\vec{m}_A\cdot\vec{M}_B]\int_0^{R_B}\int_0^{\pi}\frac{r_{bB}^2\sin\theta_{bB}}{(r_{bB}^2+r_{BA}^2+2r_{bB}r_{BA}\cos\theta_{bB})^{3/2}}d\theta_{bB}dr_{bB} \\ &-\frac{3\mu}{4}[m_{Ax}M_{Bx}+m_{Ay}M_{By}]\int_0^{R_B}\int_0^{\pi}\frac{(r_{bB}^4\sin^3\theta_{bB})}{(r_{bB}^2+r_{BA}^2+2r_{bB}r_{BA}\cos\theta_{bB})^{5/2}}d\theta_{bB}dr_{bB} \\ &-\frac{3\mu}{2}[m_{iz}M_{jz}]\int_0^{R_B}\int_0^{\pi}\frac{r_{bA}^2}{(r_{bB}^2+r_{BA}^2+2r_{bB}r_{BA}\cos\theta_{bB})^{5/2}} \\ &\quad(r_{BA}^2\sin\theta_{bB}+2r_{BA}r_{bB}\sin\theta_{bB}\cos\theta_{bB}+r_{bB}^2\sin\theta_{bB}\cos^2\theta_{bB})d\theta_{bB}dr_{bB}. \end{aligned}$$

The equation above turns to be the same as eq. 12 throught the variable change of eq. 11, such that the deduction can continue there, proving again that the magnetic interaction between these two spheres is dipolar.

VI. Conclusion

The calculation of magnetic interaction between solids might be simplified by taking symmetry properties into account from the beginning. If tensor calculus is used, the magnetostatic interaction can be written by means of an operator that depends only on space applied to magnetic moment vectors. That is particularly useful to analyse interacting bodies with shared symmetry axes. We showed that a relatively simple magnetic interaction between infinitesimal elements, with only four terms, can be applied to the interaction between solids that share an axis of rotational symmetry along the distance between their centers. We showed that two ferromagnetic spheres, arbitrarily positioned and oriented, interact as two point dipoles, provided that the magnetization inside the volume of each one of them is constant and continuous. The explicit derivation was made in progressive restriction of the properties of the volume to which it can be applied, in order to facilitate the application to bodies of different forms. We aim to extend that calculation to ellipsoids and spheres with radially-varying magnetization, useful models in the research involving magnetic nanoparticles.

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