Renormalization approach to blowup in inviscid MHD Shell Model

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Abstract

We study blowup (formation of finite-time singularity) in a shell model for magnetohydrodynamic (MHD) turbulence, showing blowup for this model and proving a criterion for it's occurance. We introduce a renormalization scheme which takes blowup time to infinity and study some basic properties of the renormalized model while proving it's consistency. Relation between the renormalized and original models is studied and we propose a method of analysis for a special solution of the renormalized model. Some numeric solutions of the models are shown.

1 Introduction

Blowup is defined as the formation of a singularity in an initially regular solution of the flow equations during a finite time.

Existence of blowup in a flow of incompressible ideal fluid lies in the forefront of modern reseach in fluid dynamics and is still an open problem even in the simplest cases, such as in the 3D Euler equations and 2D natural convection. Furthermore, blowup may be responsible for the energy cascade in developed turbulence [1, 2, 3]. However, even the numeric approach to this problem is extremely complicated, as simulations of these non-linear systems, of a great number of degrees of freedom, must be done over a large range of scales. A more detailed account on previous studies on singularity structure in ideal fluid equations can be found in [4], as well as some of the difficulties involved in the small scale simulations necessary.

Dynamic (shell) models of turbulence consist of systems of non-linear Ordinary Differential Equations (ODE's) built following the general structure of the fourier transform of the flow equations while preserving some of it's general properties, such as scaling laws and ideal invariants. Futhermore, the space of wave-vectors is discretized into concentric spheres, and to each sphere are assigned scalar variables, which are associated to the physical variables which describe the original flow and have their dynamic described by the shell model.

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This greatly reduces the number of degrees of freedom of the model while preserving many non-trivial and interesting features, such as energy and entropy cascades and anomalous turbulent spectra [5, 6], making possible very precise numerical simulations. This compels the study of shell models in search of clues to understand more clearly turbulent phenomena. But despite their simplicity, much of shell model dynamics is yet to be understood.

Being good candidates to reproduce general properties of turbulent phenomena while enabling precise simulations for system of very high Raynolds numbers, shell models can be great tools if applied to astrophysics. Simulations of magnetohydrodynamics in the early universe studying the effects of plasma viscosity on primordial magnetic fields [7] as well as studies of scaling exponents of probabilities distributions of quantities in solar flares [8] are examples of such applications.

The present work is mainly concerned with studying blowup for a simple MHD shell model and introducing a renormalization scheme for it. We study the basic properties of the renormalized model and it's relation to the original one. In section 2, incompressible MHD equations are breafly introduced. Section 3 introduces the MHD shell model studied and we prove a criterion for blowup in this model. In section 4 we define a renormalization scheme which takes blowup time to infinity, deduce the renormalized shell model and study its symmetries. Numeric solutions for the pure hydrodynamic model are given in section 5, as well as a theorem suggesting a method for their analysis. In section 6 numeric solutions are given for the MHD shell model.

2 Incompressible MHD equations

The unforced MHD equations for incompressible systems read [9]:

$$\begin{aligned}
&\frac{\partial \mathbf{v}}{\partial t} - \nu \nabla^2 \mathbf{v} = -(\mathbf{v}\cdot\nabla)\mathbf{v} + (\mathbf{b}\cdot\nabla)\mathbf{b} - \nabla p, \\
&\frac{\partial \mathbf{b}}{\partial t} - \eta \nabla^2 \mathbf{b} = \nabla \times (\mathbf{v}\times\mathbf{b}), \\
&\nabla\cdot\mathbf{v} = 0 \quad , \quad \nabla\cdot\mathbf{b} = 0,
\end{aligned} \tag{1}$$

where $\mathbf{v}$ and $\mathbf{b}$ are the velocity and induced magnetic fields, p is the total pressure, both magnetic and kinetic, while the density ρ has been taken as one. The induced magnetic field $\mathbf{b}$ was normalized by $\sqrt{4\pi\rho}$, being then measured in units of velocity. These equations follow from the Navier-Stokes equation taking into account the Lorentz force and from Maxwell equations [9]. First equation is a Cauchy momentum equation considering eletromagnetic forces and stress, and describes the momentum transport of an infinitesimal fluid volume. The second equation models magnetic field dynamics, following from Faraday's law, Ohm's law and the uniform cunductivity assumption [4]. Third and fourth equations are respectively the continuity equation, accounting for the conservation of total fluid mass, and Gauss' law for magnetism, which states the nonexistence of magnetic monopoles.

The non-linear terms on the right hand side redistribute magnetic and kinetic energy among the full range of scales of the system. When the transference of kinetic to magnetic energy exactly compensates energy dissipation caused by magnetic diffusivity, magnetic energy does not decay with time. This phenomenon is called dynamo action.

Three-dimensional systems have three ideal quadratic invariants, the total energy (E), the total correlation (C) and total magnetic helicity (H) given as follows:

$$\begin{aligned}
E &= \frac{1}{2}\int (\mathbf{v}^2 + \mathbf{b}^2) d^3x \\
C &= \int \mathbf{v}\cdot\mathbf{b} d^3x \\
H &= \int \mathbf{a}\cdot(\nabla\times\mathbf{a}) d^3x
\end{aligned}$$

where $\mathbf{a} = \nabla \times \mathbf{b}$. The reader may refer to chapter 2 of [4] for the proof of these invariances.

3 Shell Model for MHD Turbulence

Shell models are in general built by discretizing the wave vector space into concentric spheres, which raddi satisfy a geometric progression $k_n = k_0 h^n$. To each shell is assigned real or complex scalar dynamic variables analogous to the ones describing the physical flow; in the case of MHD shell models these variables are the shell velocity $v_n(t)$ and induced shell magnetic field $b_n(t)$. Considering that the equations of the shell model must have the same structure as the MHD equations in the Fourier space, these equations must have the general form

$$\begin{aligned}
\frac{dv_n}{dt} &= k_n \mathcal{B}_n + \mathcal{C}_n + \mathcal{F}_n, \\
\frac{db_n}{dt} &= k_n \mathcal{D}_n + \mathcal{E}_n + \mathcal{G}_n,
\end{aligned}$$

where $\mathcal{B}_n$ and $\mathcal{D}_n$ are quadratic nonlinear coupling terms; $\mathcal{C}_n$ and $\mathcal{E}_n$ are dissipative terms; $\mathcal{F}_n$ and $\mathcal{G}_n$ are forcing terms. All these terms are chosen in order to preserve scale invariance, specific ideal invariants and other properties of the MHD equations.

Let us take the unforced shell model for MHD turbulence modified by Gloaguen *et al.* [10] from the mixed Obukhov-Novikov hydrodynamic shell model as

$$\begin{aligned}
\frac{dv_n}{dt} &= Ak_n[v_{n-1}^2 - b_{n-1}^2 - h(v_n v_{n+1} - b_n b_{n+1})] + \\
&\quad + Bk_n[v_{n-1}v_n - b_{n-1}b_n - h(v_{n+1}^2 - b_{n+1}^2)] - \nu k_n^2 v_n, \qquad (2) \\
\frac{db_n}{dt} &= Ak_{n+1}[v_{n+1}b_n - v_n b_{n+1}] + Bk_n[v_n b_{n-1} - v_{n-1}b_n] - \eta k_n^2 b_n,
\end{aligned}$$

where $k_n = k_0 h^n$, ν is the kinematic viscosity, η is the magnetic diffusivity and A and B are arbitrary coupling coefficients. Usually, one takes $h = 2$. This system is based on the restriction to real variables, interaction only between nearest neighboors and ideal conservation of total energy and total correlation.

We are concerned with the uniparametric analysis of the inviscid model that follows from the choice $\nu = \eta = 0$, $B = 1$ and $A = \epsilon$, written as follows:

$$\begin{aligned}\frac{dv_n}{dt} &= k_n[\epsilon(v_{n-1}^2 - b_{n-1}^2) + v_{n-1}v_n - b_{n-1}b_n] \\ &- k_{n+1}[v_{n+1}^2 - b_{n+1}^2 + \epsilon(v_n v_{n+1} - b_n b_{n+1})], \\ \frac{db_n}{dt} &= \epsilon k_{n+1}[v_{n+1}b_n - v_n b_{n+1}] + k_n[v_n b_{n-1} - v_{n-1}b_n].\end{aligned} \tag{3}$$

The ideal invariants of our system are the total energy and the cross-correlation,

$$E = \tfrac{1}{2}\sum(u_n^2 + b_n^2)\ , \qquad C = \sum u_n b_n\ .$$

In the study of blowups in the model (3), i.e. the development of singularities in initialy regular solutions of our model in finite time, the following definitions of norms are most useful,

$$\begin{aligned}\|v\|_1 &= \left(\sum k_n^2 v_n^2\right)^{1/2}, \\ \|v\|_{1,\infty} &= \sup_n k_n |v_n|\,.\end{aligned} \tag{4}$$

Regular solutions of (3) are such that:

$$\|v\|_1 + \|b\|_1 < \infty\ . \tag{5}$$

Local existence of solutions satisfying the above condition can be proved using the Picard theorem in the same way as done in [11] for the inviscid Sabra shell model (a complex model for 3D MHD turbulence, considering nearest and next-nearest neighbours and imposing all three ideal invariants [8]), making the natural modifications due to the difference in the nonlinear therms.

Formation of blowup at $t = t_c$ implies that

$$\sup_{0\le t<t_c} \left(\|v\|_1 + \|b\|_1\right) = \infty. \tag{6}$$

The following theorem serves as a blowup criterion for model (3).

Theorem 1 *Let $v_n(t)$ and $b_n(t)$ be a smooth solution of* (3) *satisfying the condition* (5) *for $0 \le t < t_c$, where t_c is the maximal time of existence for such solution.*
Then, either $t_c = \infty$ or

$$\int_0^{t_c} \|v\|_{1,\infty}\, dt = \infty. \tag{7}$$

Proof: If (7) is satisfied, it follows that $\|v\|_{1,\infty}$ is unbounded for $0 \le t < t_c$ and (6) is satisfied, making (7) a sufficient condition for blowup. Let us show that it is also a necessary condition. Using the definitions (4) and equations (3) we find the relation:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\|v\|_1^2 + \|b\|_1^2\right) &= \sum k_n^2 u_n \frac{du_n}{dt} + \sum k_n^2 b_n \frac{db_n}{dt} \\ &= \sum k_n^3 v_n [\epsilon\left(v_{n+1}^2 - b_{n+1}^2 - (v_n v_{n-1} - b_n b_{n-1})\right) \\ &+ v_n v_{n+1} - b_n b_{n+1} - h\left(v_{n-1}^2 - b_{n-1}^2\right)] \\ &+ \sum k_n^3 b_n \left[h\epsilon\left(v_{n-1}b_n - v_n b_{n-1}\right) + v_n b_{n+1} - v_{n+1} b_n\right].\end{aligned} \tag{8}$$

Using the triangular and Cauchy-Schwarz inequalities as well as $k_n |v_n| \leq \|v\|_{1,\infty}$ for any shell number n, we can find the following inequality for some constant D:

$$\frac{d}{dt}\left(\|v\|_1^2 + \|b\|_1^2\right) < D\,\|v\|_{1,\infty}\left(\|v\|_1^2 + \|b\|_1^2\right). \tag{9}$$

From the use of the Grownwall inequality [12] we can find an upper bound for the sum of the squared norms:

$$\left(\|v\|_1^2 + \|b\|_1^2\right)_{t=t_c} < \left(\|v\|_1^2 + \|b\|_1^2\right)_{t=0} \exp\left(D\int_0^{t_c} \|v\|_{1,\infty}\,dt\right) \tag{10}$$

Showing that (7) is necessary for (6).

4 Renormalization of the system

We introduce a renormalization scheme, analogous to the one suggested by Dombre and Gilson [13] for the Obukhov-Novikov model [14, 15], with the purpose of moving t_c to infinity in the new system and making standard dynamical system methods available.

Definition Let τ be the renormalized time defined implicitly by

$$t = \int_0^{\tau} \exp\left(-\int_0^{\tau'} R(\tau'')d\tau''\right) d\tau' \tag{11}$$

We thus define the renormalized shell speed u_n and renormalized induced shell magnetic field β_n as

$$\begin{aligned} u_n &= \exp\left(-\int_0^{\tau} R(\tau')d\tau'\right) k_n v_n, \\ \beta_n &= \exp\left(-\int_0^{\tau} R(\tau')d\tau'\right) k_n b_n. \end{aligned} \tag{12}$$

The system of equations that describe the temporal evolution of the renormalized model can be easily obtained by differentiating (12) with respect to τ, using the definition of $t(\tau)$ given by (11) and the original system (3). Our renormalized model is thus given by:

$$\frac{du_n}{d\tau} = -R(\tau)u_n + P_n \quad , \qquad\qquad \frac{d\beta_n}{d\tau} = -R(\tau)\beta_n + Q_n \tag{13}$$

where

$$\begin{aligned} P_n &= \epsilon(h^2(u_{n-1}^2 - \beta_{n-1}^2) - u_n u_{n+1} + \beta_n \beta_{n+1}) \\ &\quad + h(u_{n-1}u_n - \beta_{n-1}\beta_n) - h^{-1}(u_{n+1}^2 - \beta_{n+1}^2), \\ Q_n &= \epsilon(u_{n+1}\beta_n - u_n\beta_{n+1}) + h(u_n\beta_{n-1} - u_{n-1}\beta_n). \end{aligned} \tag{14}$$

We can determine the function $R(\tau)$ by imposing conservation of "energy" in the renormalized system (13),

$$\frac{1}{2}\frac{d}{d\tau}\sum u_n^2 + \beta_n^2 = \sum [u_n P_n + \beta_n Q_n] - R\sum [u_n^2 + \beta_n^2] = 0.$$

Which can be satisfied choosing

$$R(\tau) = \frac{\sum u_n P_n + \beta_n Q_n}{\sum u_n^2 + \beta_n^2} \tag{15}$$

so that $\sum (u_n^2 + \beta_n^2) = C$ is conserved.

From (12) at $\tau = 0$ and (4) for initial condition of finite norm, we have

$$\sum u_n^2 = \sum k_n^2 v_n^2 = \|v\|_1^2 < \infty.$$

From the same arguments for β_n it follows that $C < \infty$.

We now verify that the function (15) is well defined for any nontrivial solution, concerning ourselves only with such solutions in what follows.

Lemma 2 *Nontrivial solution u_n and β_n of the renormalized system* (13) *exists globaly for $0 \leq \tau < \infty$ and is related by* (11) *and* (12) *to the solution v_n and b_n of the original system* (3) *for $t < t_c$, where t_c is the blowup time from Theorem* 1.

Proof: Since we have constructed the renormalized system (13) from system (3) by defining (12), it suffices only to show that (15) is well defined and that any $\tau \geq 0$ corresponds to $t < t_c$.

From the definition of the constant C above, it follows that $|u_n| < C^{1/2}$ and $|\beta_n| < C^{1/2}$. Since $C < \infty$, we need only consider the numerator in the definition of $R(\tau)$. Substitution of P_n and Q_n given by (13) leads to

$$\begin{aligned}\sum (u_n P_n + \beta_n Q_n) = \sum & u_n[\epsilon(h^2(u_{n-1}^2 - \beta_{n-1}^2) - u_n u_{n+1} + \beta_n \beta_{n+1}) \\ & + h(u_{n-1}u_n - \beta_{n-1}\beta_n) - h^{-1}(u_{n+1}^2 - \beta_{n+1}^2)] \\ & + \sum \beta_n[\epsilon(u_{n+1}\beta_n - u_n\beta_{n+1}) + h(u_n\beta_{n-1} - u_{n-1}\beta_n)]\end{aligned}$$

all the twelve terms above can be bounded in a similar way, as done bellow for the first term: $\left|\epsilon h^2 u_{n-1}^2\right| \leq |\epsilon| h^2 C$.

As such, $|R(\tau)| < \infty$ for all $\tau \geq 0$.

From the definitions (12) and that $|u_n| < C^{1/2}$ we have $|k_n v_n(t)| < \infty$, i.e. $\|v\|_{1,\infty} < \infty$, for any t corresponding to $0 \leq \tau < \infty$. By *Theorem* 1 we have that $t < t_c$.

Next, we shall study the symmetries of the renormalized system (13) and how they are related to the symmetries of the original system (3).

It follows from the usual analysis that the renormalized system has the following symmetries:
(S.R.1) $\tau \to \tau/a$, $u_n \to a u_n$, $\beta_n \to a\beta_n$ for arbitrary real constant a;
(S.R.2) $\tau \to \tau - \tau_0$ for arbitrary real constant τ_0;
(S.R.3) $u_n \to u_{n+1}$, $\beta_n \to \beta_{n+1}$

Lemma 3 *The symmetries (S.R.1)-(S.R.3) of the renormalized system* (13) *are aquivalent to the following symmetries of the original system* (3)*:*
(S.N.1) $t \to t/a$, $v_n \to a v_n$, $b_n \to a b_n$
for arbitrary real constant a;
(S.N.2) $t \to t/a - t_0$, $v_n \to a v_n$, $b_n \to a b_n$, where both a and t_0 are constants uniquely determined by τ_0 in (S.R.2);
(S.N.3) $v_n \to h v_{n+1}$, $b_n \to h b_{n+1}$

Solutions of the renormalized system (13) correspond to solution of the original system (3) by definitions (11) and (12).

Here we prove only symmetry (S.N.2), which is the most complicated. The other symmetries are proven with similar arguments.

Proof of (S.N.2):

Let $\hat{\tau} = \tau - \tau_0$, $\hat{u_n}(\hat{\tau}) = u_n(\tau)$, $\hat{\beta_n}(\hat{\tau}) = \beta_n(\tau)$. It follows that $\hat{R}(\hat{\tau}) = R(\tau) = R(\hat{\tau} + \tau_0)$.

From defenition (11):

$$\hat{t} = \int_0^{\hat{\tau}} \exp\left(-\int_0^{\tau'} \hat{R}(\tau'')d\tau''\right) d\tau' = \int_0^{\tau-\tau_0} \exp\left(-\int_0^{\tau'} R(\tau''+\tau_0)d\tau''\right) d\tau'$$

$$= \int_{\tau_0}^{\tau} \exp\left(-\int_{\tau_0}^{\xi'} R(\xi'')d\xi''\right) d\xi'$$

$$= \frac{\int_0^{\tau} \exp\left(-\int_0^{\tau'} R(\tau'')d\tau''\right) d\tau'}{\exp\left(-\int_0^{\tau_0} R(\tau')d\tau'\right)} - \int_0^{\tau_0} \exp\left(-\int_0^{\tau'} R(\tau'')d\tau''\right) d\tau'$$

$$= \frac{t}{a} - t_0$$

where the subtitution $\xi = \tau - \tau_0$ has been breafly used and

$$a = \exp\left(-\int_0^{\tau_0} R(\tau'' d\tau'')\right) \quad , \qquad t_0 = \int_0^{\tau_0} \exp\left(-\int_0^{\tau'} R(\tau'')d\tau''\right) d\tau'$$

In a similar manner:

$$\hat{v}_n(\hat{\tau}) = \exp\left(\int_0^{\hat{\tau}} \hat{R}(\tau')d\tau'\right) k_n^{-1}\ \hat{u}_n(\hat{\tau}) = \exp\left(\int_0^{\tau-\tau_0} R(\tau'+\tau_0)d\tau'\right) k_n^{-1} u_n(\tau)$$

$$= \exp\left(\int_{\tau_0}^{\tau} R(\xi')d\xi'\right) k_n^{-1} u_n(\tau) = a v_n(t)$$

Simmetry for $b_n(t)$ follows in exactly the same manner.

Blowup in the original system can be understood from analysis of limiting solutions of the renormalized system. In this work we are concerned with the study of self-similar and periodic solutions of system (13).

5 Pure hydrodynamic model

For $b_n \equiv 0$ our system reduces to:

$$\frac{dv_n}{dt} = k_n[\epsilon(v_{n-1}^2 - h v_n v_{n+1}) + v_{n-1}v_n - h v_{n+1}^2] \tag{16}$$

which is associated by definitions (11) and (12) to the system:

$$\begin{aligned} \frac{du_n}{d\tau} &= -R(\tau)u_n + P_n \\ P_n &= \epsilon(h^2 u_{n-1}^2 - u_n u_{n+1}) + h u_{n-1}u_n - h^{-1}u_{n+1}^2 \end{aligned} \tag{17}$$

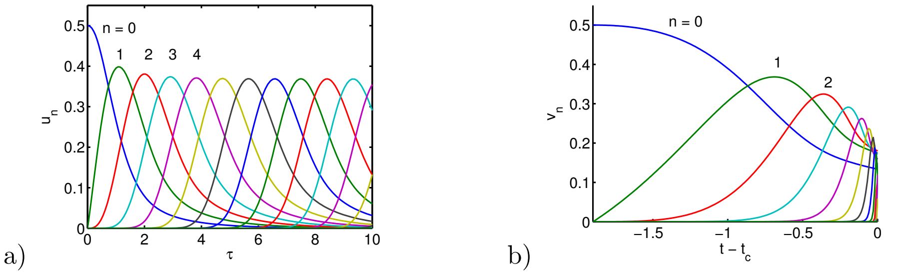

Figure 1: Blowup for the inviscid O.N. hydrodynamic shell model for $\epsilon = 0.5$: a) soliton for renormalized variable $u_n(\tau)$; b) numerical solution for variable $v_n(t)$. $t_c = 1.89$ is equilant to $\tau \to \infty$.

For ϵ between $[-10, -1]$ and $[0.5, 10]$ we have numericaly found only traveling wave asymptotic solutions for (17), which were first observed in [13]. In the following numerical solutions presented we have made the choices $k_0 = 1$, $h = 2$. Figure 1 shows solution for $\epsilon = 0.5$.

For large τ, asymptotic solutions are solitary travelling waves:

$$u_n(\tau) = aU(n - a\tau) \tag{18}$$

which travels between shells towards large n with constant positive speed a. Note that a in (18) is related to symmetry $(S.R.1)$. Function $U(\xi) \to 0$ as $\xi \to \pm\infty$.

The following theorem, modified from the similar result in [13], gives self-similar solutions for the original variables $v_n(t)$ based on the solutions found for the renormalized variables $u_n(\tau)$. Note that similar solutions for shell velocities have been found by [16, 17, 18, 19].

Theorem 4 *Taking $a = 1$ in (18), let us define the scalling exponent*

$$y = \frac{1}{\log h} \int_0^{1/a} R(\tau) d\tau \tag{19}$$

and the function

$$V(t - t_c) = \exp\left(\int_0^{\tau} R(\tau) d\tau\right) U(-\tau) \tag{20}$$

where τ is related to t by (11) and $R(\tau)$ is given by (15).

If $y > 0$, then solution $v_n(t)$ related to (18), for arbitrary posivite constant a, is given by

$$v_n(t) = ak_n^{y-1} V(ak_n^y (t - t_c)) \tag{21}$$

where the blowup time $t_c < \infty$ is given by

$$t_c = \int_0^{\infty} \exp\left(-\int_0^{\tau'} R(\tau'') d\tau''\right) d\tau' \tag{22}$$

Proof: From (3) we have that:

$$t - t_c \mapsto \frac{t - t_c}{a}, \qquad \text{then} \qquad v_n \mapsto a v_n$$

showing that the dependence of a in both (18) and (21) is due to the same symetry. As such, we can make the choice $a = 1$ in the rest of the proof without loss of generality.

First, let us show that integral (22) converges. From (18) and (15), we conclude that $R(\tau)$ must be periodic with period $1/a = 1$. As such, from definition (19), a constant D can be found satisfying the inequality

$$\int_0^\tau R(\tau')d\tau' > D + \tau y \log h > 0$$

which, from defintion of t_c gives the desired result for every positive y

$$t_c < \int_0^\infty \exp\left(-D - \tau y \log h\right) < \infty$$

From (19), definition of k_n for $k_0 = 1$ and periodicity of $R(\tau)$, for any τ we arrive at:

$$k_n^y = \exp\left(\int_\tau^{\tau+n} R(\tau')d\tau'\right) \tag{23}$$

Let us study time t' correspondent to $\tau + n$. Using definitions (11) and (22)

$$\begin{aligned} t_c - t' &= \int_{\tau+n}^\infty \exp\left(-\int_0^{\tau'} R(\tau'')d\tau''\right) d\tau' = \int_\tau^\infty \exp\left(-\int_0^{\hat\tau+n} R(\tau'')d\tau''\right) d\hat\tau \\ &= \int_\tau^\infty \exp\left(-\int_0^{\hat\tau} R(\tau'')d\tau'' - \int_{\hat\tau}^{\hat\tau+n} R(\tau'')d\tau''\right) d\hat\tau \end{aligned}$$

where change of variables $\tau' = \hat\tau + n$ was made. Using (23) we arrive at

$$t_c - t' = k_n^{-y}(t_c - t) \tag{24}$$

Similarly, using (12), (23), (18) and definition (20)

$$\begin{aligned} v_n(t') &= k_n^{-1} \exp\left(\int_0^{\tau+n} R(\tau')d\tau'\right) u_n(\tau + n) \\ &= k_n^{y-1} \exp\left(\int_0^{\tau} R(\tau')d\tau'\right) U(-\tau) = k_n^{y-1} V(t - t_c) \end{aligned}$$

Using (24) we arrive at the intended equation for $a = 1$

$$v_n(t) = k_n^{y-1} V(k_n^y(t - t_c))$$

Note that the function $V(\xi)$ and the scaling exponent y do not depend on initial conditions. As a result, asymptotic solutions of the form (21) are universal.

6 MHD model

Let us return to the MHD models (3) and (13) with nonzero shell magnetic fields.

We present numerical solutions for the initial shell model (3) showing blowup as well as numerical solutions to the renormalized system (13). All simulations are carried out under the same values for the initial conditions, which are determined for the renormalized system from initial conditions for the shell model and renormalization scheme presented in (11) and (12); the initial conditions used are: $v_0(0) = 0.5$, $v_1(0) = 0.15$, $b_0(0) = 0.3$ and $b_1(0) = 0.01$. Only the parameter ϵ is varied.

6.1 Self-similar solution

For $\epsilon = 0.5$ we have found self-similar solution for the renormalized shell speed, shown in Fig.2; blowup happens at $t_c = 1.7482$.

We observe a traveling wave solution for renormalized shell velocities, simi-

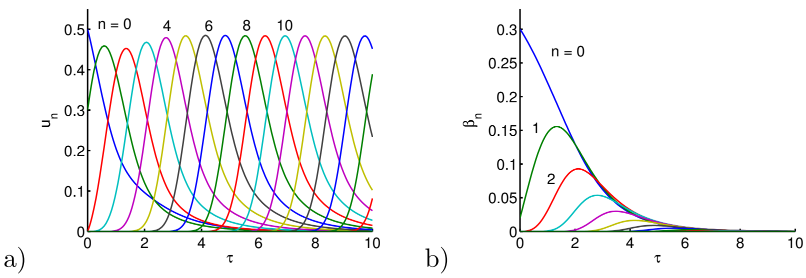

Figure 2: Numerical solutions for (13) for (a) renormalized shell speed $u_n(\tau)$ and (b) renormalized induced shell magnetic field $\beta_n(\tau)$ for $\epsilon = 0.5$.

tar to the ones shown in Fig. 1a, while the renormalized induced magnetic field vanishes rapidly, indicating transformation of "magnetic energy" to "kinetic energy".

The corresponding numeric solutions to the original shell model (3) are shown in Fig.3. Note that the general behaviour of the shell speed is very similar to the one presented for the hydrodynamic shell model in Section 5, and that blowup does not seem to happen for the induced magnetic field. Observe that, as system approaches blowup, magnetic energy increases monotonically.

6.2 Periodic solution

We have numerically found periodic solutions for the induced magnetic field for $\epsilon = -1.45$, shown in Fig.4b. For this value of ϵ we have calculated $t_c = 2.4761$.

We observe that the solution for the renormalized shell velocity is a traveling wave while the solution for the renormalized induced magnetic field is given by a slowly decaying pulsating wave. The considerably small decaying rate of the induced shell magnetic field amplitudes suggests the possibility of renormalized solutions analogous to the dynamo effect.

a) b)

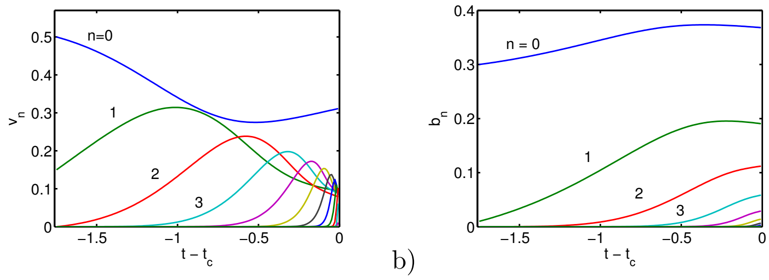

Figure 3: Numerical solutions for (3) for (a) shell speed $v_n(t)$ and (b) induced shell magnetic field $b_n(t)$ for $\epsilon = 0.5$. Blowup happens at $t_c = 1.7482$.

a) b) c)

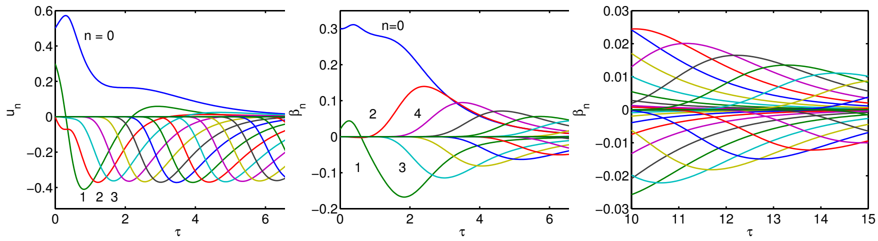

Figure 4: Numerical solutions of system (13) for (a) renormalized shell speed $u_n(\tau)$ and (b, c) renormalized induced shell magnetic field $\beta_n(\tau)$ for $\epsilon = -1.45$ at different intervals of τ.

a) b)

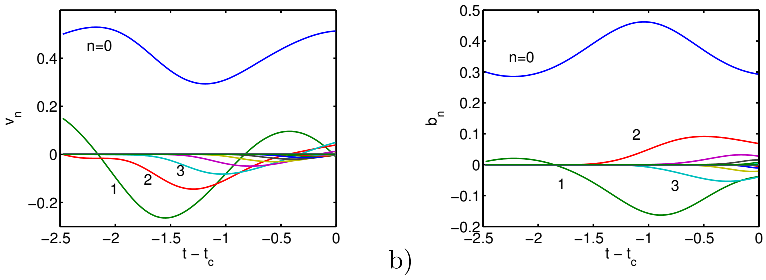

Figure 5: Numerical solutions for (3) for (a) shell speed $v_n(t)$ and (b) induced shell magnetic field $b_n(t)$ for $\epsilon = -1.45$. Blowup happens at $t_c = 2.4761$.

Fig. 5 shows numeric solution to the original shell model. We observe the same general behaviour shown in Fig. 4, suggesting the possibility of finding asymptotic solutions to the initial system (3) from analysys of solutions for the renormalized system (13), as done in Section 5 and [20].

7 Conclusion

Starting from the unforced inviscid case of the shell model of magnetic turbulence proposed in [10, 6], we numerically studied it's blowup and followed the path proposed in [20], proving a new analytic criterion for blow-up in this system. We proposed a renormalization scheme in the spirit of [13, 20] and carried out the study of the relations between the renormalized and original systems. Based on this study, a method for obtaining asymptotic solutions for the pure hydrodynamic model (16) is proposed as done by [20] and we present some numeric solutions supporting the possibility of finding a similar result for our studied model.

As a future work we expect to prove a method similar to Theorem 4 for the analysis of systems (3) and (13), as well as carry on this analysis numerically.

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