

The one-dimensional Fuzzy Ising Model

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Abstract

This work uses Fuzzy Sets Theory to develop analytical tools in order to guide the construction of the fuzzy phase diagram associated to the one-dimensional spin- $\frac{1}{2}$ Ising model subject to an external magnetic field, taking into account uncertainties presents in both exchange parameter J and magnetic field h . After that, by utilizing the fuzzy phase diagram just constructed, the consequences arising from those uncertain parameters on graphs of a few thermodynamics quantities such as magnetization and entropy are analyzed.

Keywords and phrases: Fuzzy sets, Ising Model, Fuzzy Phase Diagram, Thermodynamics.

1 Introduction

Ising-like chains [1] are amongst the most important models of the Statistical Physics as well as Condensed Matter Physics. Such an importance lies not only in the fact of its wide application to several phenomena, but also as a suitable toy model for testing more general chains. The Ising model is also applied in economy, social sciences, biology, biomembranes among other areas [2, 3, 4]. Therefore a generalization from this model can be very important for different knowledge areas.

In turn, Fuzzy Sets Theory has provided methods and algorithms where uncertain, vague or ambiguous description or reasoning are required, as occur in the fields of artificial intelligence, economics, pattern recognition (fuzzy clustering), medicine, ecology and theory of information [5].

In [6] the Ising model was studied by replacing all the numbers (Ising-spins) by fuzzy numbers and all operations for their correspondents fuzzy operations. In the same article, it was shown that the defuzzified partition function by the center of gravity method engender a partition function of a network of soft-spin scalar (Ginzburg-Landau), and in addition there are other possible models for mapping the network.

In article [7] was proposed a description of evolution of fuzzy systems with uncertainty, and has been shown that this dynamics has the form of a Hamiltonian on the extended state space composed of physical and information components, if used the t-min-norm. It was also shown that if the product t-norm is used,

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the dynamics in a continuous universe is similar to the dynamics of stochastic, but with the probability distribution exchanged by the density of possibilities.

The present work uses another strategy to put uncertainties into the classical model. As an alternative to [6], it was chosen to preserve the entire fundamental structure –i.e, that concerning Quantum Mechanics– and add uncertainties on the measurable parameters presents, namely, exchange energy J and external magnetic field h . From this, it is proposed a way to build the fuzzy phase diagram for this model and analyze the emergent properties generated by the fuzzyfication process. Additionally the thermodynamic properties are analyzed through confrontation between the physical quantities obtained from the fuzzyfication and defuzzification process with the correspondents classical (standard) quantities.

The outline of this work is as following: in section 2 a quick rememoration of the standard thermodynamic of the concerned model is performed; in section 3, and at appendix, are presented some of the main definitions and results of Fuzzy Sets Theory, in particular results, mainly the **Theorem 3.1** and the **Proposition 3.2**, which will be utilized later on; section 4 discusses the fuzzy thermodynamic of the concerned model, exposed firstly in subsection 4.1 in which it was constructed the fuzzy phase diagram with the support of the mentioned theorems, and after, in subsection 4.2, constituted by the fuzzy graphs of two thermodynamic quantities; finally the conclusions presented in sec. 5.

2 The Model and standard Thermodynamic

The hamiltonian of the one-dimensional spin- $\frac{1}{2}$ Ising Model is given by $H = \sum_{i=1}^N H(\sigma_i, \sigma_{i+1})$, where

$$H(\sigma_i, \sigma_{i+1}) = -J\sigma_i\sigma_{i+1} - \frac{h}{2}(\sigma_i + \sigma_{i+1}), \quad (1)$$

is the interaction energy of a plaquette composed by two Ising-spins, whereas N is the number of sites, $\sigma_i = \pm 1$, J is the exchange constant representing the interaction force beetwen two spins and h symbolizes the external magnetic field.

To study the thermodynamic is necessary the largest eigenvalue Λ_+ of the transfer matrix (further details can be seen in [8]), given by

$$\Lambda_+ = \frac{1}{2} \left(Z_{(1,1)} + Z_{(-1,-1)} \right) + \frac{1}{2} \sqrt{\left(Z_{(1,1)} - Z_{(-1,-1)} \right)^2 + 4Z_{(1,-1)}Z_{(-1,1)}} \quad (2)$$

where

$$Z_{(\sigma_1\sigma_2)} = \exp(\beta J(\sigma_1\sigma_2) + \frac{\beta h}{2}(\sigma_1 + \sigma_2)) \quad (3)$$

and $\beta = \frac{1}{kT}$ (in this expression, k is the Boltzmann constant and T is the temperature). So, the free energy of the system in thermodynamic limit reads

$$\mathcal{W} = -\frac{1}{\beta} \lim_{N \rightarrow \infty} \frac{\ln(\Lambda_+^N)}{N} = -\frac{1}{\beta} \ln(\Lambda_+). \quad (4)$$

From this, one can study the standard thermodynamic quantities of the model: entropy ($S = -\frac{\partial \mathcal{W}}{\partial T}$), specific heat ($C = T \frac{\partial S}{\partial T}$), and magnetization ($M = -\frac{\partial \mathcal{W}}{\partial h}$).

The building of the phase diagram for this system requires the knowledge of the ground state energies of a plaquette of it, summarized by (1). Thus, the energies in which we are interested are

$$\begin{aligned} E_1(J, h) &= -J - h, \\ E_2(J, h) &= J, \\ E_3(J, h) &= -J + h. \end{aligned} \tag{5}$$

To build it, we must select the lowest value of the three energies above, namely E_1 or E_2 or E_3 , for each value of J and h . After this we plot as in fig. 1. From this figure it is noted that the intersections between the ground state energies, $E_1 \cap E_2$ and $E_1 \cap E_3$ and $E_2 \cap E_3$, are contained in standard phase diagram, forming the boundaries between one energy and another. Selecting it, one obtains as in fig. 2.

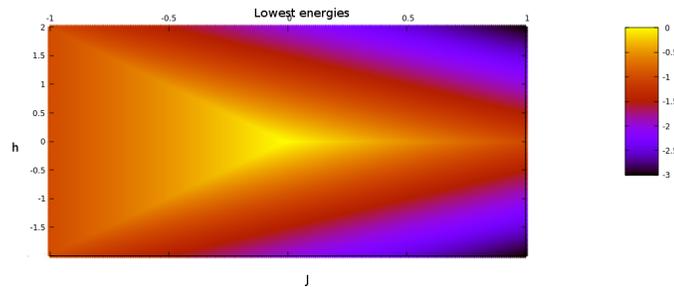


Figure 1: Graph which presents the lowest energies for each pair (J, h) builded from (5). The colors represent values of energy.

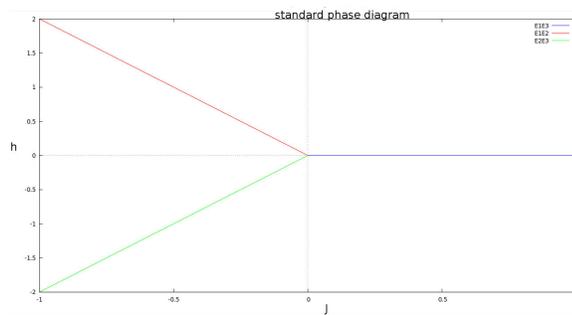


Figure 2: Phase diagram builded from the intersections between ground state energies, displayed in (5).

The analysis by way of graphs of the thermodynamics properties aforementioned is guided by the phase diagram constructed from energies above (see fig. 2). This work intends perform a similar procedure, just by replacing the energies in (5) and the thermodynamic properties by its fuzzy counterparts. However is necessary know how to do that. It is the purpose of next section providing this knowledge.

3 Fuzzy Sets Theory

In general terms, each set can be assigned a membership function, f , which will be responsible for assigning to each element of a previously defined universe set one of the following two values: 0 or 1. If the value of the function in the element is 0, then this element does not belong to referred set; otherwise it belongs.

However, in 1965 Zadeh [9] published a work in which a new class of sets was defined. In this, whose sets are called fuzzy sets, the membership function is allowed to assign any value within the range $[0, 1]$. A good example of a fuzzy set could be a cloud since the interior points closer to the center belong “more” to the cloud than those in its diffuse boundary [5].

There are many useful concepts in Fuzzy Sets Theory, such as α -cuts, fuzzy numbers, and the *Extension Principle* stated by Zadeh for application in fuzzy functions, but all are presented in appendix along with the demonstration of following theorem, a very important result for the developments to come. In it $\mathfrak{F}(\mathbb{R})$ designates the set of all fuzzy numbers.

Theorem 3.1. Let $F : [\alpha, \beta] \times [\gamma, \delta] \rightarrow \mathfrak{F}(\mathbb{R})$ be a function that associates a pair of real numbers to a fuzzy number, defined by

$$F(x, y; a, r_a, b, r_b) = Ag(x) + Bh(y), \quad (6)$$

such that $g : [\alpha, \beta] \rightarrow \mathbb{R}$ and $h : [\gamma, \delta] \rightarrow \mathbb{R}$ are monotonic functions and A and B are triangular fuzzy numbers, respectively centered in a and b , with r_a and r_b uncertainties given by

$$A(t) = \begin{cases} \frac{t+r_a-a}{r_a}, & \text{if } t \in [a-r_a, a] \\ \frac{r_a+a-t}{r_a}, & \text{if } t \in [a, a+r_a] \end{cases}$$

and

$$B(t) = \begin{cases} \frac{t+r_b-b}{r_b}, & \text{if } t \in [b-r_b, b] \\ \frac{r_b+b-t}{r_b}, & \text{if } t \in [b, b+r_b]. \end{cases}$$

i) If $g(x), h(y) \geq 0$, $\forall x \in [\alpha, \beta]$ and $\forall y \in [\gamma, \delta]$, then

$$F(x, y; a, r_a, b, r_b)(t) = \begin{cases} \frac{t+g(x)(a-r_a)+h(y)(b-r_b)}{r_ag(x)+r_bh(y)}, & t \in [g(x)(a-r_a)+h(y)(b-r_b), \\ & ag(x)+bh(y)] \\ \frac{g(x)(a+r_a)+h(y)(b+r_b)-t}{r_ag(x)+r_bh(y)}, & t \in [ag(x)+bh(y), \\ & g(x)(a+r_a)+h(y)(b+r_b)]; \end{cases}$$

ii) If $g(x) \leq 0$ and $h(y) \geq 0$ $\forall x \in [\alpha, \beta]$ and $\forall y \in [\gamma, \delta]$, then

$$F(x, y; a, r_a, b, r_b)(t) = \begin{cases} \frac{t+|g(x)|(a+r_a)-h(y)(b-r_b)}{r_a|g(x)|+r_bh(y)}, & t \in [-|g(x)|(a+r_a)+h(y)(b-r_b), \\ & -a|g(x)|+bh(y)] \\ \frac{|g(x)|(a-r_a)+h(y)(b+r_b)-t}{r_ag(x)+r_bh(y)}, & t \in [-a|g(x)|+bh(y), \\ & -|g(x)|(a-r_a)+h(y)(b+r_b)] \end{cases}$$

iii) If $g(x), h(y) \leq 0$, $\forall x \in [\alpha, \beta]$ and $\forall y \in [\gamma, \delta]$, then

$$F(x, y; a, r_a, b, r_b)(t) = \begin{cases} \frac{t + |g(x)|(a+r_a) + |h(y)|(b+r_b)}{r_a|g(x)| + r_b|h(y)|}, & t \in [-|g(x)|(a+r_a) - |h(y)|(b-r_b), \\ & -a|g(x)| - b|h(y)|] \\ \frac{|g(x)|(a-r_a) + |h(y)|(b+r_b) - t}{r_a g(x) + r_b h(y)}, & t \in [-a|g(x)| - b|h(y)|, \\ & -|g(x)|(a-r_a) + |h(y)|(b-r_b)] \end{cases}$$

In the following it is enunciated a result that could be stated as an observation; due to its simplicity, the demonstration will be omitted. Nevertheless the appearance, it will play a key role in guiding the interpretation of the results to come.

Proposition 3.2. Let A be a triangular fuzzy number of uncertainty denoted by r such that $A(a) = 1$ and $A(x), A(y) > 0$ such that $|x - y| < r$. We claim that the membership is an “almostness” relation, i.e,

- the membership of x with respect to A is equal to membership of a with respect to X ;
- the memberships of x to Y and of y to X are equal each other;

where X and Y are triangular fuzzy numbers centered in x and in y both of uncertainty r .

Remark 1. We can to do the following interpretation, much more suggestive:

- i) Of course a real number is almost itself with degree of success equal to 1;
- ii) If a real number is almost another, then this latter is almost the former with the same degree of success;
- iii) Under the conditions of the proposition, if x is almost a and a is almost y then x is almost y .

4 Fuzzy Thermodynamic

This section is divided in two subsections. In the first one, will be constructed the fuzzy phase diagram for the Ising model. Graphs of a few “fuzzified” thermodynamic properties are displayed in order of examine the behavior of it after the fuzzification process as well as to ascertain the existence of a plausible linking between what the fuzzy phase diagram display and what the fuzzified thermodynamic properties show.

4.1 Fuzzy Phase Diagram

According to [5], the uncertainty, vagueness and ambiguity which are inherent to Fuzzy Mathematics can be attributed to either: (a) presence of subjectivity in the variables chosen or to, (b) lack of complete information of the system in consideration or an incomplete definition of the system. In this work the latter situation is present, since there is no complete and definite information about the values assumed by the magnetic field and the exchange parameter, respectively h and J .

To construct the fuzzy phase diagram, the variables J and h , in (5), will be replaced by their fuzzy counterparts. This can be done by setting

$$\mathbf{J}(J)(t) = S_J(t)J \quad (7)$$

and

$$\mathbf{h}(h)(t) = S_h(t)h, \quad (8)$$

where

$$S_J(t) = \begin{cases} \frac{t + \frac{r_J}{|J|} - 1}{\frac{r_J}{|J|}}, & t \in [1 - \frac{r_J}{|J|}, 1] \\ \frac{\frac{r_J}{|J|} + 1 - t}{\frac{r_J}{|J|}}, & t \in [1, 1 + \frac{r_J}{|J|}] \end{cases}, \quad S_h(t) = \begin{cases} \frac{t + \frac{r_h}{|h|} - 1}{\frac{r_h}{|h|}}, & t \in [1 - \frac{r_h}{|h|}, 1] \\ \frac{\frac{r_h}{|h|} + 1 - t}{\frac{r_h}{|h|}}, & t \in [1, 1 + \frac{r_h}{|h|}]. \end{cases}$$

Note that the functions S_s (“S” of special) were designed in such a way to preserve the uncertainties of J and of h .

Thus the symbolic expressions for the energies stay

$$\begin{aligned} \mathbf{E}_1(J, h) &= -\mathbf{J} - \mathbf{h}, \\ \mathbf{E}_2(J, h) &= \mathbf{J}, \\ \mathbf{E}_3(J, h) &= -\mathbf{J} + \mathbf{h}. \end{aligned} \quad (9)$$

Following the standard procedure, the fuzzy phase diagram is builded from fuzzy operation *min* of the fuzzy energies above, given by $\text{Min}(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3)$ (see **Proposition A.3**). However with the aid of the **Proposition A.4** and **Theorem 3.1**, one note that among the intersections $\mathbf{E}_1 \cap \mathbf{E}_2$, $\mathbf{E}_1 \cap \mathbf{E}_3$ and $\mathbf{E}_2 \cap \mathbf{E}_3$:

- For $J \leq 0$ and $h \geq 0$, only

$$\mathbf{E}_1 \cap \mathbf{E}_2 \subset \text{Min}(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3)$$

- For $J, h \leq 0$, only

$$\mathbf{E}_2 \cap \mathbf{E}_3 \subset \text{Min}(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3)$$

- For $J \geq 0$ and $h \leq 0$, only

$$(\mathbf{E}_1 \cap \mathbf{E}_3) \cup (\mathbf{E}_2 \cap \mathbf{E}_3) \subset \text{Min}(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3)$$

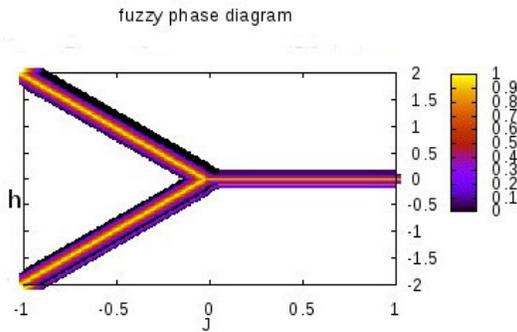
- For $J, h \geq 0$, only

$$(\mathbf{E}_1 \cap \mathbf{E}_3) \cup (\mathbf{E}_1 \cap \mathbf{E}_2) \subset \text{Min}(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3)$$

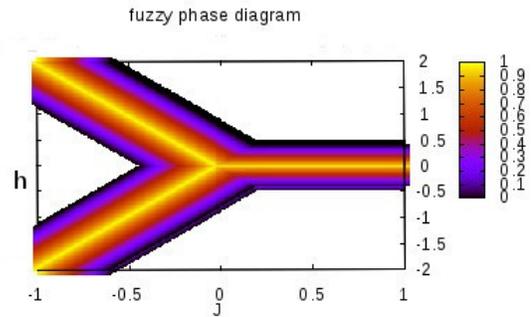
The result above shows that the intersections among the ground state energies are contained in the lowest values of these energies. Because a phase diagram is made up by the intersections among the lowest energies, only referred intersections will be used for build it.

Doing so, the correspondent expressions in fuzzy form will be searched, namely, $\mathbf{E}_1 \cap \mathbf{E}_3$, $\mathbf{E}_1 \cap \mathbf{E}_2$ and $\mathbf{E}_2 \cap \mathbf{E}_3$ and so join it. The calculations of these expressions are easy but tedious, since is necessary deal with the α -cuts of them, but can be performed with help of the **Theorem 3.1** together with his

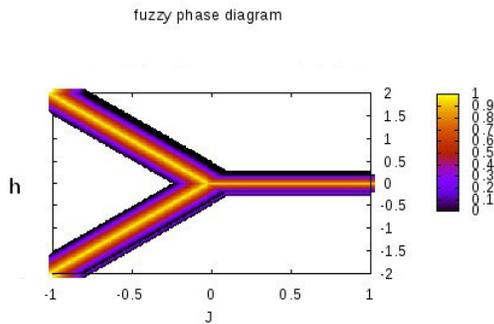
demonstration. After doing them, one finds the functions whose graphs are displayed below. The colors show the membership degree of the points with respect to the associated classical phase diagram ($r_h = r_J = 0$) displayed in fig. 2.



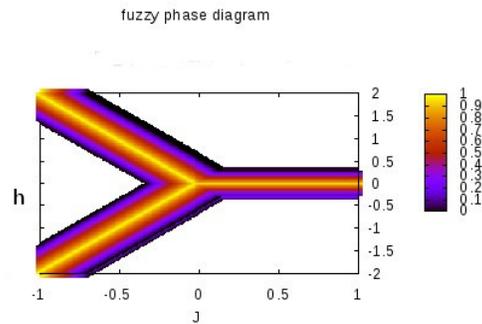
(a) $r_J = 0.1, r_h = 0.$



(b) $r_J = 0.4, r_h = 0.$



(c) $r_J = 0, r_h = 0.4$



(d) $r_J = 0.2, r_h = 0.2$

4.2 Thermodynamic Quantities

Now are presented some fuzzy graphs of two thermodynamic quantities, namely, entropy and magnetization. For the construction of them, it was taken the classical (or standard) expressions of these quantities and we fuzzified it by means of the Extension Principle (**Axiom 4**). Of course that to do this one needs before provide, according to **Axiom 2**, the expressions which will define the fuzzy number associated to variable (J or h) which are intended to “fuzzificate”. In such a case, will be used triangular fuzzy numbers, as those displayed by **Theorem 3.1**.

Also will be shown the associated classical graphs displayed together with the graphs resulting of defuzzification process, in which the center-of-gravity (COG) defuzzification method was utilized. The aim of showing a defuzzified graph is not take it by the true representation of a property of the fuzzy Ising model (whose role is played by the fuzzy graph), but as a way of provide insights on the fuzzified system.

The aim is to ascertain whether the colorful regions presented by the fuzzy phase diagram (i.e, the graphs themselves) coincide with transition zones. For this, was first fixed one variable (J or h) and started with a point away of the plausible zone and compared the fuzzy graph with its correspondent classical version at low temperatures. After this, penetrated into the plausible transition zone and choosed two points on the same line (vertical or horizontal): one immediately before and other immediately after of point on the classical graph (membership equal to one). The two fuzzy grahs was compared between them and among correspondents classical versions. Finally, a point away of the plausible transition zone was taken.

4.2.1 Fuzzy Entropy

If any positive J is identified as ferromagnetic phase and any negative J as antiferromagnetic phase then, according to interpretation of **Proposition 3.2**, may to exist zones in which such phases are not well-defined. More precisely, let $J_1 < 0$ and $J_2 > 0$ be numbers such that $|J_1 - J_2| < r$, where r is the uncertainty on the exchange energy J . So J_1 represent the antiferromagnetic phase whereas J_2 represent the ferromagnetic phase. Hence follows of referred proposition that the antiferromagnetic phase is almost ferromagnetic, and conversely; what lead us to say that these two phase are not well-delimited.

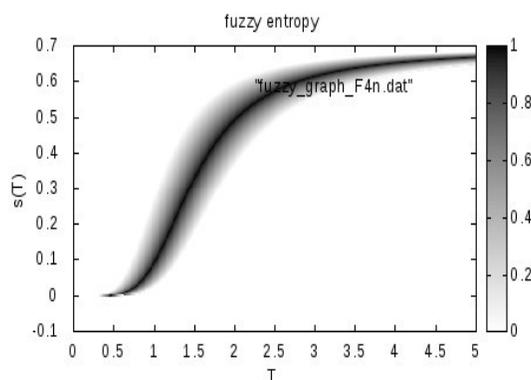


Figure 3: Fuzzy entropy in which $r_J = 0.4$, $r_h = 0$, $J = 1$ and $h = 0.4$. The colors show the membership degree of points with respect to the associated classical graph.

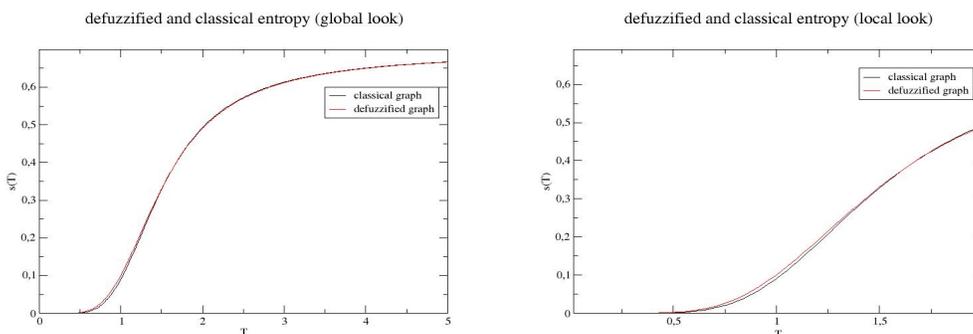


Figure 4: On the left side, the classical graph with $J = 1$ and $h = 0.4$, overlapped to defuzzification of the fuzzy graph above (see fig.3). On the right side, the behavior at low temperatures. Note that the starting point of the larger membership line of the fuzzy graph is the same that one of its classical version.

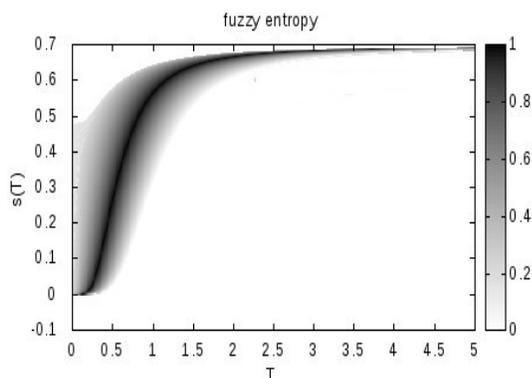


Figure 5: Fuzzy entropy in which $r_J = 0.4$, $r_h = 0$, $J = 0.15$ and $h = 0.4$. The colors show the membership degree of points with respect to the associated classical graph.

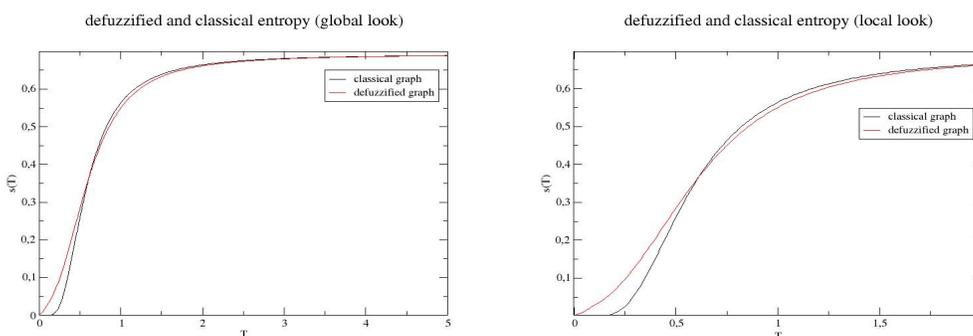


Figure 6: On the left side, the classical graph in which $J = 0.2$, $h = 0.4$ overlapped to defuzzification of the fuzzy graph above (see fig.5). On the right side, the behavior at low temperatures. The analysis of these graphs must be made together with next graphs (see figs 7 and 8). Note that given a fuzzy graph (e.g. that of fig. 5), the range at near zero temperatures comprehend the starting point of the classical graph associated to the other fuzzy graph (in this case, that one of fig.7).

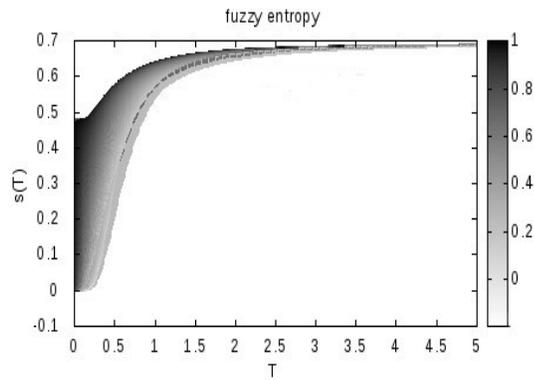


Figure 7: Fuzzy entropy in which $r_J = 0.4$, $r_h = 0$, $J = -0.15$ and $h = 0.4$. Should be noted that both the fuzzy graphs (this and the previous) has a global behaviour very similar. The colors show the membership degree of points with respect to the associated classical graph.

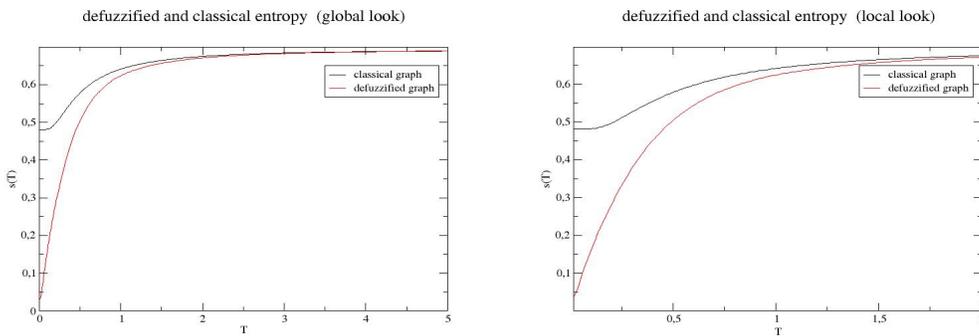


Figure 8: On the left side, the classical graph with $J = -0.15$ and $h = 0.4$, overlapped to defuzzification of the fuzzy graph above (see fig. 7). The analysis is the same that one in previous case.

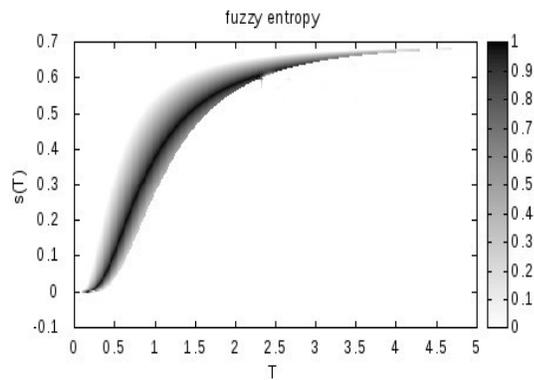


Figure 9: Fuzzy entropy in which $r_J = 0.4$, $r_h = 0$, $J = -1$ and $h = 0.4$. The colors show the membership degree of points with respect to the associated classical graph.

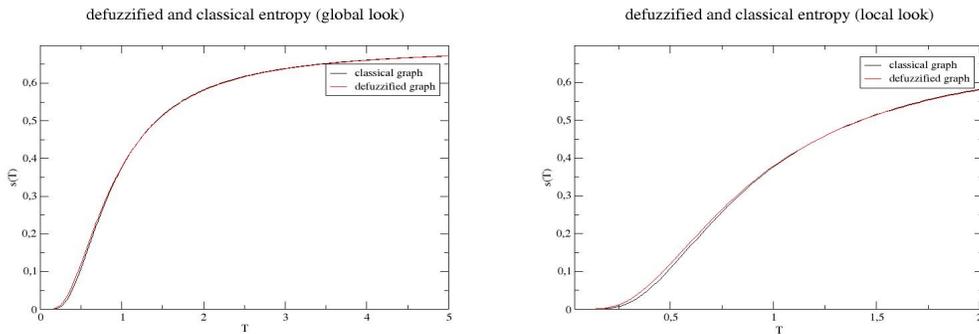


Figure 10: On the left side, the classical graph with $J = -1$ and $h = 0.4$, overlapped to defuzzification of the fuzzy graph above (see fig.9). Note that, in the fuzzy graph, the starting point of the larger membership line is the same that one of its classical version.

If the fuzzy graph in fig. 3 and the classical graph in fig. 4 are compared each other, and the fuzzy graph in fig. 9 and the classical graph in fig. 10 also are compared each other, it will be noted that the fuzzy graphs are very similar to their classical counterparts. This fact is strengthened by the perfect overlapping displayed between the classical graphs and the defuzzified graphs.

On the other hand, the fuzzy graphs lying in figures 5 and 7 are morphologically similar each other, with special attention to temperatures near zero. This happens due to proximity between J s. It may be necessary to note that the differences between the defuzzified graphs and the correspondents classical versions displayed, respectively, in figures 6 and 8, are remarkable.

4.2.2 Fuzzy Magnetization

An analysis will now be developed taking into account only changes in horizontal direction of the fuzzy phase diagram. More precisely, it will be fixed a value of J and leave the other variable, namely h , moving. Hence the fuzzy phase diagram (c) will be used. Often the analysis of concerned fuzzy phase diagram will have a classical phase diagram as reference. The phase diagram (a) display three approximately well-delimited regions, what allow us to use it in practical analysis following. For such a phase diagram (i.e, a classical phase diagram) one see that for a fixed J the transition between a region and the adjacent to it is “abrupt”, in sense that it occurs through an only point. This is the most important characteristic of a classical phase diagram that will be used here. In the following, again the whole analysis will be performed with the help of **Proposition 3.2**. In all graphs below $J = -0.5$. One start with a point away of the likely transition zone (note that in this moment we are referring to the fuzzy phase diagram (c)), namely, $h = 0.5$. The correspondent graphs to this point are presents in figures 11 and 12. In this case, note that the behavior of the fuzzy graph is similar to its associated classical graph, at least at low temperatures, that is what matter. Now compare the fuzzy graphs of the figures 13, 15 and 17 each other. The behavior of the three fuzzy graphs near zero are very similar, in agreement with the **Proposition 3.2**, because the uncertainty is $r_h = 0.4$. Therefore these three fuzzy graphs, considered in sequence, suggest us that the transition in progress is “smooth”, in opposition to the classic case which, for $J = -0.5$, shall occur on

the point $h = 1$, as is easy realize. Note, still, that the differences between the defuzzified graphs and the respective classical versions are remarkable. Lastly, because the fuzzy graph of fig. 19 is away of transition zone, it is similar to its correspondent classical version.

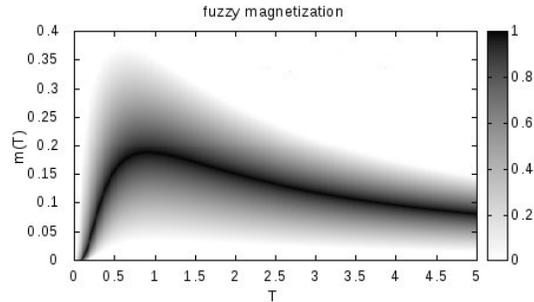


Figure 11: Fuzzy magnetization in which $r_J = 0$, $r_h = 0.4$, $J = -0.5$ and $h = 0.5$. The colors show the membership degree of points with respect to the associated classical graph.

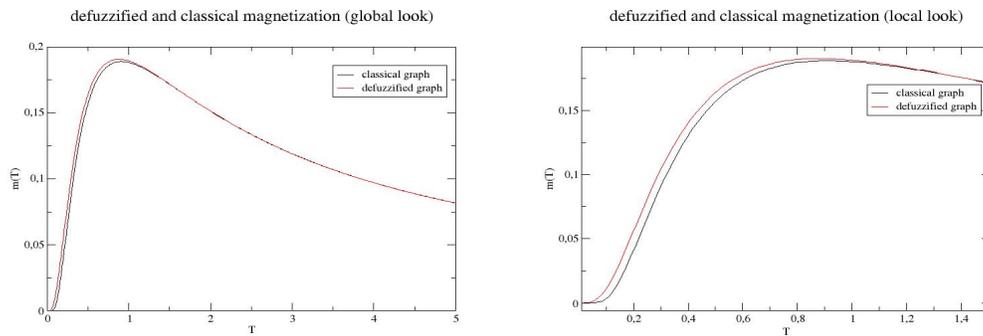


Figure 12: On the right side, the correspondent classical graph with $J = -0.5$ and $h = 0.5$, overlapped to defuzzification of the fuzzy graph above (see fig.11). On the left side, the behavior at low temperatures. Note that because the fuzzy graph is away of the classical transition point (which is $h = 1$), the starting points of both of the graphs in this figure are the same.

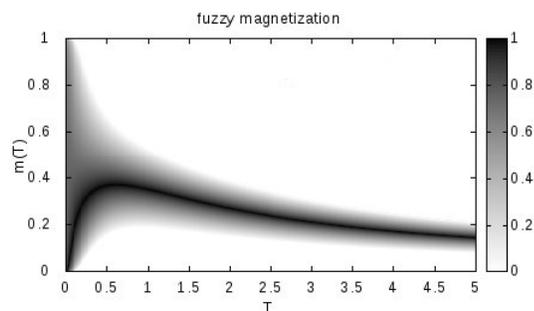


Figure 13: Fuzzy magnetization in which $r_J = 0$, $r_h = 0.4$, $J = -0.5$ and $h = 0.9$. The colors show the membership degree of points with respect to the associated classical graph.

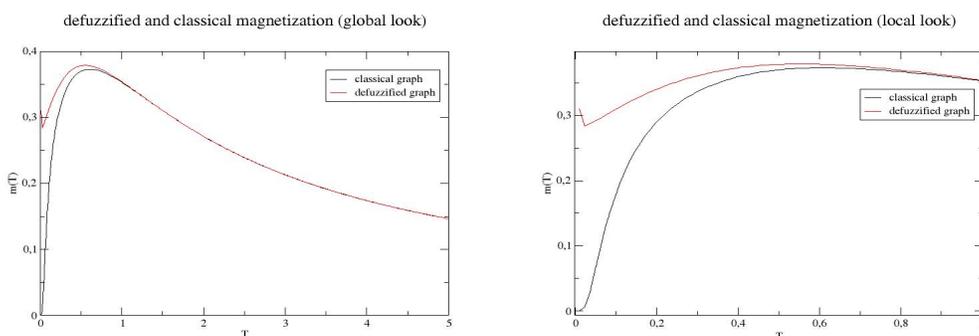


Figure 14: On the right side, the classical graph with $J = -0.5$, $h = 0.5$ overlapped to defuzzification of the fuzzy graph above (see fig.13). This fuzzy graph together with the two next fuzzy graphs (namely, fig.15 and 17) must be analysed at same time. Note that the distance of the concerned point ($h = 0.9$) to the classical transition point ($h = 1$) is less than the uncertainty r_h , so we are into transition zone.

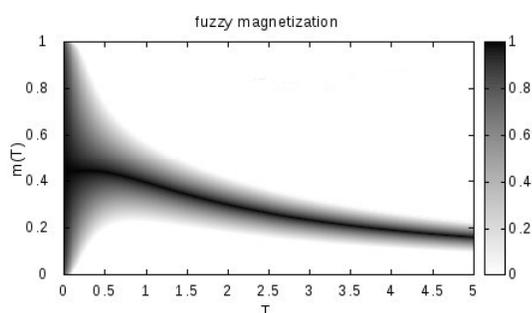


Figure 15: Fuzzy magnetization in which $r_J = 0$, $r_h = 0.4$, $J = -0.5$ and $h = 1$. The colors show the membership degree of points with respect to the associated classical graph.

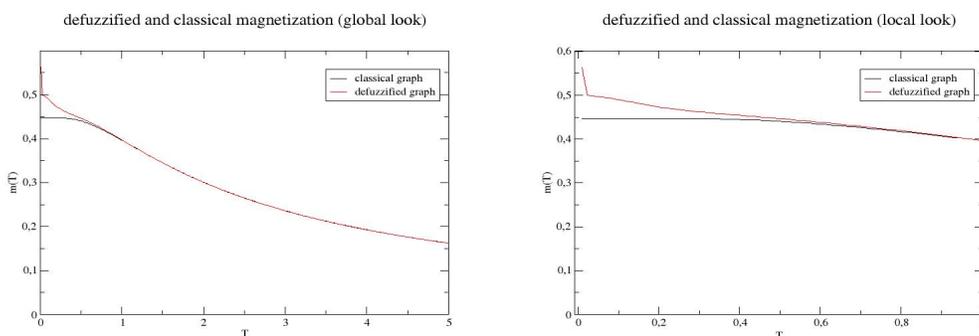


Figure 16: On the right side, the classical graph in which $J = -0.5$, and $h = 1$, overlapped to defuzzification of the fuzzy graph above (see fig.15). Now we are on the classical transition point.

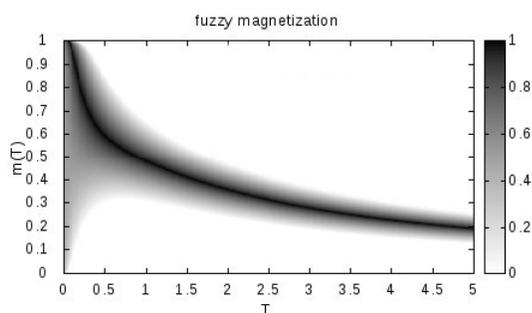


Figure 17: Fuzzy magnetization in which $r_J = 0$, $r_h = 0.4$, $J = -0.5$ and $h = 1.2$. Because we are still into transition zone, the behavior of this graph near zero tend to be similar to that one presented by two last fuzzy graphs. The colors show the membership degree of points with respect to the associated classical graph.

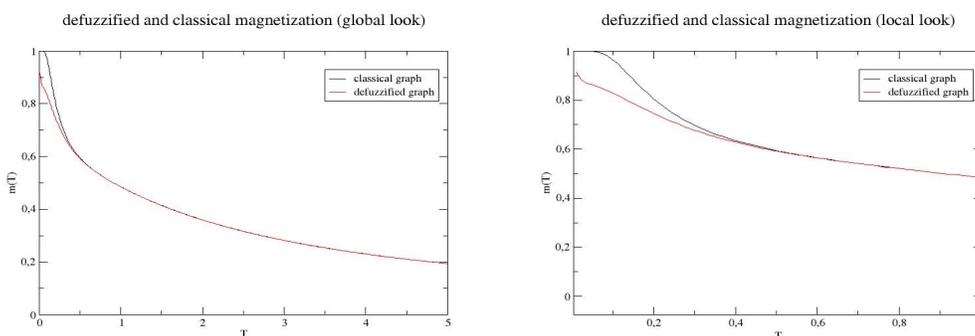


Figure 18: On the right side, the classical graph in which $J = -0.5$, and $h = 1.2$, overlapped to defuzzification of the fuzzy graph above (see fig.17).

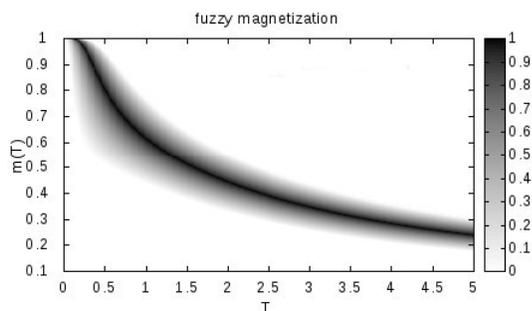


Figure 19: Fuzzy magnetization in which $r_J = 0$, $r_h = 0.4$, $J = -0.5$ and $h = 1.5$. The colors show the membership degree of points with respect to the associated classical graph.

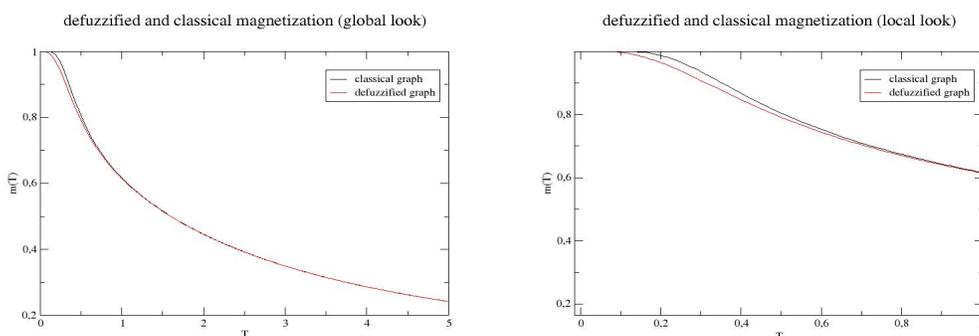


Figure 20: On the right side, the classical graph with $J = -0.5$, and $h = 1.5$, overlapped to defuzzification of the fuzzy graph above (see fig.19). Because we are out of the transition zone, the behavior of fuzzy graph near zero is similar to the presented by its associated classical graph in the same region.

5 Conclusions

One can list at least three interesting points, necessarily not independent of each other:

- i) Firstly, it was noticed that the connection between the phase diagram and the behavior of the thermodynamic properties in addition to be retained is generalized after to be done the fuzzification;
- ii) For fixed J (or h), after the fuzzification the transition zones leave to be single points for becomes a set of them. The outcome of this is an apparently more “smooth” transition;
- iii) And finally, it was noted that the more salient discrepancies between the defuzzified and classical graphs become manifest at low temperatures when the measurable parameters J and h takes values in transition zones.

One can conclude by noting two points which are not entirely irrelevant about the analytical method for the construction of some fuzzy phase diagrams developed by this work: at same time teach to fuzzificate a class of functions, namely monotonic functions, it show how analytically to deal with them.

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A Appendix: Fuzzy Sets Theory

In this section are presented some definitions and results concerning fuzzy sets. For this, just for sake of clarity, firstly are stated some axioms designed to treat indistinctly both classical sets and fuzzy sets. Because are widely discussed in literature, no reference will be made except in special cases. For an approach much more verbose, see [10].

Axiom 1. There exists a set.

Remark 2. It will be called it *universe set* and denoted by U .

Defintion 1. [membership function] It is said that f is a membership function if its domain is U and its image is contained in $[0, 1]$; $f : U \rightarrow [0, 1]$.

Definition 2. [support] It is called *support* of $f : U \rightarrow [0, 1]$ the collection of elements of U such that the image of each of which under f is nonzero ($f(x) > 0$).

Axiom 2. A set is entirely defined by its membership function f .

Remark 3. Due to this axiom will be employed the same symbol to represent both the set and its associated membership function.

Axiom 3. The membership function of U is $U : U \rightarrow [0, 1]$, such that $U(x) = 1, \quad \forall x \in U$.

Definition 3. [subset] A is a subset of B , and denoted $A \subset B$, if

$$A(x) \leq B(x), \quad \forall x \in U.$$

Remark 4. It's easy to conclude that every set is subset of U . It will be assumed this tacitly hereafter.

Definition 4. [intersection] If A and B are sets, then the intersection between them is the set

$$(A \cap B)(x) = \min \{A(x), B(x)\}, \quad x \in U.$$

Definition 5. [union] If A and B are sets, then the union between them is the set

$$(A \cup B)(x) = \max \{A(x), B(x)\}, \quad x \in U.$$

Definition 6. [α -cut] Let A be a set. The α -cut of A , A^α , where $\alpha \in [0, 1]$, is the set defined as follows:

$$A^\alpha(x) = \begin{cases} 1, & \text{if } A(x) \geq \alpha \\ 0, & \text{otherwise} \end{cases}$$

for all $x \in U$.

Proposition A.1. [10] If A and B are sets, then

i) $(A \cup B)^\alpha = A^\alpha \cup B^\alpha,$

ii) $(A \cap B)^\alpha = A^\alpha \cap B^\alpha.$

Proposition A.2. [10] $A \subset B \Leftrightarrow A^\alpha \subset B^\alpha, \quad \forall \alpha \in [0, 1].$

The next axiom which stated is the renowned *Extension Principle* of Zadeh (see [9]). From it, the thermodynamic properties in a strictly computational process will be *fuzzificate*.

Axiom 4. [Extension Principle] Let $f : M \rightarrow N$ be a function such that $M = M_1 \times M_2 \times \dots \times M_k$, and let $A = A_1 \times A_2 \times \dots \times A_k \subset M$ and $B \subset N$ be sets. The induced sets by f from the latter has the following memberships functions:

$$f^{-1}(B) : U \rightarrow [0, 1],$$

defined by

$$f^{-1}(B)(x) = B(f(x)),$$

and

$$f(A) : U \rightarrow [0, 1],$$

defined by

$$f(A)(y) := \sup_{y=f(x_1, x_2, \dots, x_k)} \min \{A_1(x_1), A_2(x_2), \dots, A_k(x_k)\}.$$

Definition 7. [fuzzy number] Let $A : \mathbb{R} \rightarrow [0, 1]$ be a set. It is a *fuzzy number* if it satisfy the following conditions:

- i) There exists x such that $A(x) = 1$,
- ii) A^α is compact for all $\alpha \in (0, 1]$,
- iii) the support of A is bounded.

The last axiom able us to extent the four arithmetic operations as well as the *min* and *max* operations to the fuzzy numbers.

Definition 8. If A and B are fuzzy numbers and $*$ any of the arithmetic operations (such as sum, subtraction, multiplication and division), then

$$(A * B)(z) = \sup_{z=x*y} \min \{A(x), B(y)\},$$

$$\text{Min}(A, B)(z) = \sup_{z=\min\{x,y\}} \min \{A(x), B(y)\},$$

$$\text{Max}(A, B)(z) = \sup_{z=\max\{x,y\}} \min \{A(x), B(y)\}.$$

Proposition A.3. [11] Let A and B continuous fuzzy numbers such that $A^\alpha = [a_1(\alpha), b_1(\alpha)]$ e $B^\alpha = [a_2(\alpha), b_2(\alpha)]$. Then

$$\text{Min}(A, B)^\alpha = [\min \{a_1(\alpha), a_2(\alpha)\}, \min \{b_1(\alpha), b_2(\alpha)\}],$$

$$\text{Max}(A, B)^\alpha = [\max \{a_1(\alpha), a_2(\alpha)\}, \max \{b_1(\alpha), b_2(\alpha)\}].$$

The next result will be very useful for the decision on how will be constructed the fuzzy phase diagram. The correspondent demonstration can be found in [11] in an indirect way.

Proposition A.4. [11] Let A and B be any fuzzy sets. Then $A \cap B \subset \text{Min}(A, B)$

In the following are stated two lemmas which will provide support for the demonstration of the main result of this work withing the framework of Fuzzy Sets Theory.

Lemma A.1. [12] If $[a, b]$ and $[c, d]$ are intervals of real line, then

$$\begin{aligned} [a, b] + [c, d] &= [a + c, b + d] \\ [a, b] - [c, d] &= [a - d, b - c]. \end{aligned}$$

Lemma A.2. [10] If A e B are continuous fuzzy numbers, then

$$(A * B)^\alpha = A^\alpha * B^\alpha,$$

for $\alpha \in [0, 1]$.

Lemma A.3. [10] If A is a set, then

$$A = \bigcup_{\alpha \in [0, 1]} \alpha A^\alpha,$$

that is,

$$A = \text{sup}_{\alpha \in [0, 1]} \{ \alpha A^\alpha \}.$$

Finally will be stated and demonstrated the most important result of this appendix.

Theorem A.8. Let $F : [\alpha, \beta] \times [\gamma, \delta] \rightarrow \mathfrak{F}(\mathbb{R})$ be a function that associates a pair of real numbers to a fuzzy number, defined by

$$F(x, y; a, r_a, b, r_b) = Ag(x) + Bh(y), \tag{10}$$

such that $g : [\alpha, \beta] \rightarrow \mathbb{R}$ and $h : [\gamma, \delta] \rightarrow \mathbb{R}$ are monotonic functions and A and B are triangular fuzzy numbers, respectively centered in a and b , with r_a and r_b uncertainties given by

$$A(t) = \begin{cases} \frac{t+r_a-a}{r_a}, & \text{if } t \in [a-r_a, a] \\ \frac{r_a+a-t}{r_a}, & \text{if } t \in [a, a+r_a] \end{cases}$$

and

$$B(t) = \begin{cases} \frac{t+r_b-b}{r_b}, & \text{if } t \in [b-r_b, b] \\ \frac{r_b+b-t}{r_b}, & \text{if } t \in [b, b+r_b]. \end{cases}$$

i) If $g(x), h(y) \geq 0, \quad \forall x \in [\alpha, \beta] \quad \text{and} \quad \forall y \in [\gamma, \delta]$, then

$$F(x, y; a, r_a, b, r_b)(t) = \begin{cases} \frac{t+g(x)(a-r_a)+h(y)(b-r_b)}{r_ag(x)+r_bh(y)}, & t \in [g(x)(a-r_a) + h(y)(b-r_b), \\ & ag(x) + bh(y)] \\ \frac{g(x)(a+r_a)+h(y)(b+r_b)-t}{r_ag(x)+r_bh(y)}, & t \in [ag(x) + bh(y), \\ & g(x)(a+r_a) + h(y)(b+r_b)]; \end{cases}$$

ii) If $g(x) \leq 0$ and $h(y) \geq 0 \quad \forall x \in [\alpha, \beta]$ and $\forall y \in [\gamma, \delta]$, then

$$F(x, y; a, r_a, b, r_b)(t) = \begin{cases} \frac{t + |g(x)|(a+r_a) - h(y)(b-r_b)}{r_a|g(x)| + r_b h(y)}, & t \in [-|g(x)|(a+r_a) + h(y)(b-r_b), \\ & -a|g(x)| + bh(x)] \\ \frac{|g(x)|(a-r_a) + h(y)(b+r_b) - t}{r_a g(x) + r_b h(y)}, & t \in [-a|g(x)| + bh(y), \\ & -|g(x)|(a-r_a) + h(y)(b+r_b)] \end{cases}$$

iii) If $g(x), h(y) \leq 0, \quad \forall x \in [\alpha, \beta]$ and $\forall y \in [\gamma, \delta]$, then

$$F(x, y; a, r_a, b, r_b)(t) = \begin{cases} \frac{t + |g(x)|(a+r_a) + |h(y)|(b+r_b)}{r_a|g(x)| + r_b|h(y)|}, & t \in [-|g(x)|(a+r_a) - |h(y)|(b-r_b), \\ & -a|g(x)| - b|h(y)|] \\ \frac{|g(x)|(a-r_a) + |h(y)|(b+r_b) - t}{r_a g(x) + r_b h(y)}, & t \in [-a|g(x)| - b|h(y)|, \\ & -|g(x)|(a-r_a) + |h(y)|(b-r_b)] \end{cases}$$

Proof. It will be proved only the item i). The remaining proofs are analogous.

In accordance with last stated lemma, one can express any set in terms of its α -cuts. Therefore,

$$F(x, y) = \bigcup_{\alpha \in [0,1]} \alpha F^\alpha(x, y),$$

where

$$\begin{aligned} F^\alpha(x, y) &= (g(x)A + h(y)B)^\alpha \\ &= (g(x)A)^\alpha + (h(y)B)^\alpha \\ &= g(x)A^\alpha + h(y)B^\alpha. \end{aligned}$$

(11)

The last step is of very easy checking.

Firstly we will find the α -cuts:

$$A(t) = \frac{t + r_a - a}{r_a} \geq \alpha \quad \Rightarrow \quad t \geq r_a(\alpha - 1) + a,$$

$$A(t) = \frac{r_a + a - t}{r_a} \geq \alpha \quad \Rightarrow \quad t \leq r_a(1 - \alpha) + a,$$

$$\Rightarrow A^\alpha = [r_a(\alpha - 1) + a, r_a(1 - \alpha) + a].$$

$$B(t) = \frac{t + r_b - b}{r_b} \geq \alpha \quad \Rightarrow \quad t \geq r_b(\alpha - 1) + b,$$

$$B(t) = \frac{r_b + b - t}{r_b} \geq \alpha \quad \Rightarrow \quad t \leq r_b(1 - \alpha) + b,$$

$$\Rightarrow B^\alpha = [r_b(\alpha - 1) + b, r_b(1 - \alpha) + b].$$

Thus,

$$\begin{aligned} F^\alpha(x,y) &= g(x)[r_a(\alpha - 1) + a, r_a(1 - \alpha) + a] + h(y)[r_b(\alpha - 1) + b, r_b(1 - \alpha) + b] \\ &= [g(x)r_a(\alpha - 1) + g(x)a + h(x)r_b(\alpha - 1) + h(y)b, g(x)r_a(1 - \alpha) + g(x)a + \\ &\quad h(y)r_b(1 - \alpha) + h(x)b], \end{aligned} \tag{12}$$

if

$$g(x), h(y) \geq 0, \quad \forall x \in [\alpha, \beta] \quad \text{and} \quad \forall y \in [\gamma, \delta].$$

$$\begin{aligned} \underline{\alpha = 0}: & \quad [g(x)(a - r_a) + h(y)(b - r_b), g(x)(r_a + a) + h(y)(b + r_b)] \\ \underline{\alpha = 1}: & \quad [ag(x) + bh(y), ag(x) + bh(y)] \end{aligned} \tag{13}$$

First interval:

$$\begin{aligned} g(x)r_a(\alpha - 1) + ag(x) + h(y)r_b(\alpha - 1) + bh(y) &\leq t \leq ag(x) + bh(y) \\ \Rightarrow \alpha &\leq \frac{t + g(x)(a - r_a) + h(y)(b - r_b)}{r_ag(x) + r_bh(y)} \leq 1. \end{aligned}$$

We know that

$$F(x,y)(t) = \sup_{\alpha} \{ \alpha F^\alpha(x,y)(t) \};$$

$$\begin{aligned} F(x,y)^\alpha(t) &= \begin{cases} 1, & t \in [g(x)(a - r_a) - |h(y)|(b - r_b), ag(x) + bh(y)] \\ 0, & \text{otherwise} \end{cases} \\ \Rightarrow \alpha F(x,y)^\alpha(t) &= \begin{cases} \alpha, & t \in [g(x)(a - r_a) - |h(y)|(b - r_b), ag(x) + bh(y)] \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

Thus,

$$F(x,y)(t) = \frac{t + g(x)(a - r_a) + h(y)(b - r_b)}{r_ag(x) + r_bh(y)}, \tag{14}$$

if

$$t \in [g(x)(a - r_a) + h(y)(b - r_b), ag(x) + bh(y)].$$

Second interval:

$$\begin{aligned} ag(x) + bh(y) &\leq t \leq r_ag(x)(1 - \alpha) + ag(x) + r_bh(y)(1 - \alpha) + bh(x) \\ \Rightarrow \alpha &\leq \frac{(a + r_a)g(x) + (b + r_b)h(y) - t}{r_ag(x) + r_bh(y)} \leq 1. \end{aligned}$$

Therefore,

$$F(x,y)(t) = \frac{(a+r_a)g(x) + (b+r_b)h(y) - t}{r_ag(x) + r_bh(y)}, \quad (15)$$

if

$$t \in [ag(x) + bh(y), g(x)(a+r_a) + h(y)(b+r_b)].$$

□