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Similar Ways of Creating Thirdness: Kant's "Synthetic Judgments a priori" and Frege's "Thoughts" as Intermediate Cases*

[Modos Similares de Criação de Terceiridade: os "Juízos Sintéticos *a priori*" de Kant e os "Pensamentos" de Frege enquanto Casos Intermediários]

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Abstract: It is well-known that Kant and Frege offer seemingly exclusive answers to the (epistemo)logical status of equations. Expressions like "7 + 5 = 12" are *synthetic* for Kant but *analytic* for Frege. Nevertheless, Kant and Frege have a shared interest: Demonstrating the possibility of grasping a general *realm* by *science*. Kant's question is "How is *metaphysics* as science possible?" Frege answers the question "How is *logic* as science possible?" Both thinkers are convinced that a precondition for answering their questions consists in the creation of a third concept. But how? Traditionally given mutually exclusive distinctions seem to let no room for such a different third concept. The revolutionary idea is to create basic concepts as *molecules* (*patterns*, *Gestalten*) *with a characteristic inner structure* opposed to *atoms* without any inner structure. Such molecules – Kant's "synthetic judgments a priori" and Frege's "thoughts" – can be analyzed as 2-dimensionally structured intermediate cases.
Keywords: Kant. Frege. Logic. Third concept. Intermediate case.

Resumo: Sabe-se que Kant e Frege oferecem respostas aparentemente exclusivas ao *status* logico(epistemológico) de equações. Expressões do tipo "7+5=12" são sintéticas para Kant, mas analíticas para Frege. Contudo, Kant e Frege possuem um interesse comum: demonstrar a possibilidade de captar um *domínio* geral pela *ciência*. A questão de Kant é "Como a *metafísica* enquanto ciência é possível?" Frege responde à questão "Como a *lógica* enquanto ciência é possível?" Ambos os pensadores estão convencidos de que uma pré-condição para responder suas perguntas consiste na criação de um terceiro conceito. Mas como? Distinções mutuamente exclusivas tradicionalmente dadas não parecem dar espaço para um terceiro conceito tão diferente. A ideia revolucionária é criar conceitos básicos como *moléculas* (*padrões*, *formas*) *com uma estrutura interna característica*, oposta a *átomos*, sem nenhuma estrutura interna. Tais moléculas – os "juízos sintéticos *a priori* de Kant e os "pensamentos de Frege – podem ser analisados como casos intermediários estruturados em duas dimensões.
Palavras-chave: Kant. Frege. Lógica. Terceiro conceito. Caso intermediário.

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Introduction

It is well-known that Kant and Frege offer seemingly exclusive answers to the (epistemo)logical status of equations. Expressions like "7 + 5 = 12" are *synthetic* for Kant but *analytic* for Frege. Nevertheless, Kant and Frege have a shared interest: Demonstrating the possibility of grasping a general *realm* by *science*. Kant's question is "How is *metaphysics* as science possible?" Frege answers the question "How is *logic* as science possible?" Both thinkers are convinced that a precondition for answering their questions consists in the creation of a third concept. But how? Traditionally given mutually exclusive distinctions seem to let no room for such a different third concept. The revolutionary idea is to create basic concepts as *molecules* (*patterns*, *Gestalten*) *with a characteristic inner structure* opposed to *atoms* without any inner structure. Such molecules – Kant's "synthetic judgments a priori" and Frege's "thoughts" – can be created as connecting/intermediate cases. A byproduct consists in showing why both thinkers give opposite answers with respect to the status of equations like "7 + 5 = 12" as *synthetic* (Kant) and *analytic* (Frege) on the one side. But both agree that their status is *a priori* on the other side.

In a first step we explicate our concept 2-*dimensionally structured intermediate case*. It is the core concept of our *language of analysis* to interpret selected text passages written by Kant and Frege. We use this language to show that Kant's "synthetic judgments a priori" as well as Frege's "thoughts" can be reconstructed as 2-dimensionally structured intermediate cases. But there is an important difference. Kant has to answer the question "How is metaphysics as science possible?" But he did not present his scientific metaphysics, his transcendental philosophy in an advanced *theoretical* form. Frege has to answer the question "How is logic as science possible?" Unlike Kant Frege gives his own creative answer to the question "How does this logic look like?" as well. Frege does not only show that logic is possible but he also formulates his *Begriffsschrift* as a full-blooded logical system which we call *classical logic* nowadays. We have to consider *thoughts* in both respects: (Frege 1) *thoughts* between *things of the outer world* and *ideas* (things of the inner world without decisions) and (Frege 2) *thoughts* as sense of sentences comprising *the true* and *the false* as possible meanings (bivalence). To do the latter we have to sketch a two-dimensional logic. Using this logic we can offer an alternative reading of Frege's unary negation connective and make the differences between negation and negative judgments as well as Frege's critique of Kant's position explicit. Finally, we try to convince our reader that *the content stroke / the horizontal*

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stroke in Frege's *Begriffsschrift* can be expressed by an operator which takes 1-dimensional atomic arguments p and yields elementary thoughts of the form $\begin{bmatrix} p \\ \neg p \end{bmatrix}$, whereas the judgment stroke / the vertical stroke at the very beginning of each axiom or provable formula yields the opposite, the reduction of a logically 2-dimensional thought $\begin{bmatrix} A \\ \neg A \end{bmatrix}$ or its opposite thought $\begin{bmatrix} \neg A \\ A \end{bmatrix}$ to the first dimension in question: A or $\neg A$, respectively.

1. Creating third concepts by inventing (2-dimensionally) structured intermediate cases

There is an inspiring remark by Ludwig Wittgenstein in his *Philosophical Investigations*:

> A main source of our failure to understand is that we do not *command a clear view* of the use of our words.—Our grammar is lacking in this sort of perspicuity. A perspicuous representation produces just that understanding which consists in 'seeing connexions'. Hence the importance of finding and inventing *intermediate cases*.
>
> The concept of a perspicuous representation is of fundamental significance for us. It earmarks the form of account we give, the way we look at things. (Is this a 'Weltanschauung'?) (WITTGENSTEIN 1986, p. 49: PI 122)[1]

Wittgenstein characterizes an interesting aspect of his own method of philosophizing: finding and inventing *intermediate cases*. *Finding* intermediate cases seems to be related to *observation*, *empirical* evidence. *Inventing* intermediate cases appears to point to *creating theoretical* concepts. But Wittgenstein is aware that both aspects are relevant for his philosophical enterprise: By taking a closer look at the use of our everyday language we can *find* practices where we *observe* cases which we can *understand* as intermediate cases. Otherwise the philosopher can *invent* other cases by *creating* new (fictional) practices, new *language games*. The surprising fact is that Wittgenstein *denies* that creating something new that way is a *theoretical* act. It is only *giving / cre-*

[1] "Es ist eine Hauptquelle unseres Unverständnisses, daß wir den Gebrauch unserer Wörter nicht übersehen. – Unserer Grammatik fehlt es an Übersichtlichkeit. – Die übersichtliche Darstellung vermittelt das Verständnis, welches eben darin besteht, daß wir die Zusammenhänge sehen. Daher die Wichtigkeit des Findens und des Erfindens von Zwischengliedern." Der Begriff der übersichtlichen Darstellung ist für uns von grundlegender Bedeutung. Er bezeichnet unsere Darstellungsform, die Art, wie wir die Dinge sehen. (Ist dies eine Weltanschauung?) (WITTGENSTEIN 1984, p. 302: PU 122). If the quotation in the text body is an English translation from German we offer the corresponding original passage in the respective footnote in any case.

ating examples without any intention to generalize them or to find an essence in them:

> One gives examples and intends them to be taken in a particular way.—I do not, however, mean by this that he is supposed to see in those examples that common thing which I—for some reason—was unable to express; but that he is now to *employ* those examples in a particular way. Here giving examples is not an *indirect* means of explaining—in default of a better. For any general definition can be misunderstood too. The point is that *this* is how we play the game. (I mean the language-game with the word "game".) (WITTGENSTEIN, PI 71).[2]

Starting with Wittgenstein we will go in another direction: We explain the structure of 2-dimensional intermediate cases by stating *a set of frame conditions* for them. If postulating such frame conditions is an act of creation, then this act can be the first step, a precondition of creating a *formal theory* – as in the case of Frege's *thoughts* – or not – as in the case of Kant's *synthetic judgments a priori* as a core construction to create space for metaphysics as a science.

Let me mention that from my point of view developing *2-dimensionally structured intermediate cases* is not excluded by Wittgenstein. He is not interested in this activity if it is not connected with philosophical problems. Nevertheless, creating new basic concepts in a theoretical environment is at the same moment a *philosophical* activity. And creating 2-dimensionally structured intermediate cases in our *language of analysis* provides a tool to interpret philosophical texts in an inspiring way.

Frame condition 1 (realms as sets of something and their possible intermediate cases): Let us start with two possibly separated realms as sets of these structures: $\left\{\begin{matrix} A_1 \\ \vdots \\ A_n \end{matrix}\right\}$ and $\left\{\begin{matrix} B_1 \\ \vdots \\ B_m \end{matrix}\right\}$ assuming that all pairs consisting of A_i $(1 \leq i \leq n)$ and B_j $(1 \leq j \leq m)$ are totally different from each other. Then we get *a structured intermediate case* if we construct *third realms* containing at least one but not all A_i's and at least one but not all B_j's: $\left\{\begin{matrix} \vdots \\ A_i \\ \vdots \\ B_j \\ \vdots \end{matrix}\right\}$. The use of curly brac-

[2] "Man gibt Beispiele und will, daß sie in einem gewissen Sinn verstanden werden. – Aber mit diesem Ausdruck meine ich nicht: er solle nun in diesen Beispielen das Gemeinsame sehen, welches ich – aus irgend einem Grunde - nicht aussprechen konnte. Sondern: er solle diese Beispiele nun in bestimmter Weise verwenden. Das Exemplifizieren ist hier nicht ein indirektes Mittel der Erklärung, – in Ermanglung eines Bessern. Denn, mißverstanden kann auch jede allgemeine Erklärung werden. So spielen wir eben das Spiel (Ich meine das Sprachspiel mit dem Wort »Spiel«.)" (WITTGENSTEIN 1984, p. 280 f.: PU 71).

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kets indicates that the internal order does not matter yet. This intermediate case has at least something in common with $\left\{\begin{matrix} A_1 \\ \vdots \\ A_n \end{matrix}\right\}$ $(= A_i)$, something not in common with $\left\{\begin{matrix} A_1 \\ \vdots \\ A_n \end{matrix}\right\}$ $(\neq B_j)$, something in common with $\left\{\begin{matrix} B_1 \\ \vdots \\ B_n \end{matrix}\right\}$ $(= B_j)$ and something not in common with $\left\{\begin{matrix} B_1 \\ \vdots \\ B_n \end{matrix}\right\}$ $(\neq A_i)$. The vertical dots in the intermediate case can represent new C's which are different from all the A_i's and B_j's in the first two realms. Of course, there are infinitely many such structured intermediate cases. This holds even for $i = j = 1$. E.g., $\left\{\begin{matrix} A \\ B \end{matrix}\right\}$ is an intermediate between A and B. It is important that the (ontological, linguistic etc.) *status* of the entries A_i and B_j is left open. They can stand for objects, individuals, features, properties, names, relations, sentences, judgments, tones, intervals, chords, etc. The same holds for their *logical form*. It can be atomic or complex with respect to formation rules.

Why is the third realm an *intermediate* case? $\left\{\begin{matrix} A_1 \\ \vdots \\ A_n \end{matrix}\right\}$ and $\left\{\begin{matrix} B_1 \\ \vdots \\ B_m \end{matrix}\right\}$ are completely different. But with respect to the third realm we get the following minimal picture:

$$\left\{\begin{matrix} \vdots \\ A_i \\ \vdots \\ \vdots \\ \vdots \end{matrix}\right\} \begin{matrix} \square \\ = \\ \square \\ \neq \\ \square \end{matrix} \left\{\begin{matrix} \vdots \\ A_i \\ \vdots \\ B_j \\ \vdots \end{matrix}\right\} \begin{matrix} \square \\ \neq \\ \square \\ = \\ \square \end{matrix} \left\{\begin{matrix} \vdots \\ \vdots \\ \vdots \\ B_j \\ \vdots \end{matrix}\right\}.$$

The third realm connects the first two completely different realms. It gets a central / middle position without indicating any order between the two other realms. Instead of creating one intermediate case we can construct a *series*. Take this simple example:

$$\left\{\begin{matrix} \boldsymbol{A_1} \\ A_2 \end{matrix}\right\} - \left\{\begin{matrix} \boldsymbol{A_1} \\ \boldsymbol{C_1} \end{matrix}\right\} - \left\{\begin{matrix} \boldsymbol{C_1} \\ \boldsymbol{C_2} \end{matrix}\right\} - \left\{\begin{matrix} \boldsymbol{B_1} \\ \boldsymbol{C_2} \end{matrix}\right\} - \left\{\begin{matrix} \boldsymbol{B_1} \\ B_2 \end{matrix}\right\}.^{3}$$

[3] You cannot change the position of two sets in this series without getting unconnected pairs. But this is not necessarily so:

Frame condition 2 (the first two realms as pairs): We can restrict the first two realms to pairs of the forms $\left\{ \begin{array}{c} A_1 \\ A_2 \end{array} \right\}$ and $\left\{ \begin{array}{c} B_1 \\ B_2 \end{array} \right\}$. Then there are exactly four *patterns* of intermediate cases:

$$\left\{ \begin{array}{c} \vdots \\ A_1 \\ \vdots \\ B_1 \\ \vdots \end{array} \right\}, \left\{ \begin{array}{c} \vdots \\ A_1 \\ \vdots \\ B_2 \\ \vdots \end{array} \right\}, \left\{ \begin{array}{c} \vdots \\ A_2 \\ \vdots \\ B_1 \\ \vdots \end{array} \right\} \text{and} \left\{ \begin{array}{c} \vdots \\ A_2 \\ \vdots \\ B_2 \\ \vdots \end{array} \right\}.^4$$

Frame condition 3 (the third realm as a pair): We can restrict the possible third realm to a *pair* of entries. Now we can ask "What are the intermediate pairs between $\left\{ \begin{array}{c} A_1 \\ A_2 \end{array} \right\}$ and $\left\{ \begin{array}{c} B_1 \\ B_2 \end{array} \right\}$?" and we get four *solutions*: $\left\{ \begin{array}{c} A_1 \\ B_1 \end{array} \right\}$, $\left\{ \begin{array}{c} A_1 \\ B_2 \end{array} \right\}$, $\left\{ \begin{array}{c} A_2 \\ B_1 \end{array} \right\}$ and $\left\{ \begin{array}{c} A_2 \\ B_2 \end{array} \right\}$.

Frame condition 4 (ordered pairs): Instead of looking at unordered pairs we can look at *ordered* pairs of the forms $\left[\begin{array}{c} A_1 \\ A_2 \end{array} \right]$ and $\left[\begin{array}{c} B_1 \\ B_2 \end{array} \right]$. We call expressions of these forms 2-*dimensionally structured*. The upper dimension (the dimension of A_1 and B_1, respectively) is named *first* dimension. The lower dimension (the dimension of A_2 and B_2, respectively) is named *second* dimension. We can add the condition that the form of our 2-dimensionally structured intermediate case has an analogous form: $\left[\begin{array}{c} C_1 \\ C_2 \end{array} \right]$.

Frame condition 5 (coincidence of dimensions): We can require that identity and non-identity has to be realized by preserving the respective dimension. It means that the index in the first dimension is always 1 and in the second dimension always 2. Then we get exactly two solutions:

$\left\{ \begin{array}{c} \mathbf{A_1} \\ A_2 \end{array} \right\} - \left\{ \begin{array}{c} \mathbf{A_1} \\ B_1 \end{array} \right\} - \left\{ \begin{array}{c} \mathbf{A_1} \\ \mathbf{B_2} \end{array} \right\} - \left\{ \begin{array}{c} B_1 \\ \mathbf{B_2} \end{array} \right\}$ and $\left\{ \begin{array}{c} \mathbf{A_1} \\ A_2 \end{array} \right\} - \left\{ \begin{array}{c} \mathbf{A_1} \\ B_2 \end{array} \right\} - \left\{ \begin{array}{c} \mathbf{A_1} \\ \mathbf{B_1} \end{array} \right\} - \left\{ \begin{array}{c} \mathbf{B_1} \\ B_2 \end{array} \right\}$.
Let us mention a musical example: The simplest and not very convincing way to look on chords is to analyze them simply as sets of (atomic) tones. The following C-major-cadence is an illuminating example of a closed sequence: 3-tone-C-major-chord (root position) — 3-tone-F-major-chord (second inversion) — 4-tone-G-major-seventh-chord (first inversion) — back to 3-tone-C-major-chord (root position): $\left\{ \begin{array}{c} g \\ e \\ \mathbf{c} \end{array} \right\} - \left\{ \begin{array}{c} a \\ \mathbf{f} \\ \mathbf{c} \end{array} \right\} - \left\{ \begin{array}{c} \mathbf{g} \\ \mathbf{f} \\ d \\ h \end{array} \right\} - \left\{ \begin{array}{c} \mathbf{g} \\ e \\ c \end{array} \right\}$.

[4] If we consider tuples (ordered structures) we get eight patterns, because the pattern <…,A_1,…,B_1,…> is different from the pattern <…,B_1,…,A_1,…> *etc.*

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$$\begin{bmatrix} \boldsymbol{A_1} \\ A_2 \end{bmatrix} - \begin{bmatrix} \boldsymbol{A_1} \\ \boldsymbol{B_2} \end{bmatrix} - \begin{bmatrix} B_1 \\ \boldsymbol{B_2} \end{bmatrix} \text{and} \begin{bmatrix} A_1 \\ \boldsymbol{A_2} \end{bmatrix} - \begin{bmatrix} \boldsymbol{B_1} \\ \boldsymbol{A_2} \end{bmatrix} - \begin{bmatrix} \boldsymbol{B_1} \\ B_2 \end{bmatrix}$$

or by showing (non-)identity explicitly

$$\begin{bmatrix} \boldsymbol{A_1} \\ A_2 \end{bmatrix} \begin{matrix} = \\ \neq \end{matrix} \begin{bmatrix} \boldsymbol{A_1} \\ \boldsymbol{B_2} \end{bmatrix} \begin{matrix} \neq \\ = \end{matrix} \begin{bmatrix} B_1 \\ \boldsymbol{B_2} \end{bmatrix} \text{and} \begin{bmatrix} A_1 \\ \boldsymbol{A_2} \end{bmatrix} \begin{matrix} \neq \\ = \end{matrix} \begin{bmatrix} \boldsymbol{B_1} \\ \boldsymbol{A_2} \end{bmatrix} \begin{matrix} = \\ \neq \end{matrix} \begin{bmatrix} \boldsymbol{B_1} \\ B_2 \end{bmatrix}.$$

In this structure we call $\begin{bmatrix} A_1 \\ B_2 \end{bmatrix}$ and $\begin{bmatrix} B_1 \\ A_2 \end{bmatrix}$ 2-*dimensionally structured intermediate cases*. But what about

$$\begin{bmatrix} \boldsymbol{A_1} \\ B_2 \end{bmatrix} \begin{matrix} = \\ \neq \end{matrix} \begin{bmatrix} \boldsymbol{A_1} \\ \boldsymbol{A_2} \end{bmatrix} \begin{matrix} \neq \\ = \end{matrix} \begin{bmatrix} B_1 \\ \boldsymbol{A_2} \end{bmatrix} \text{and} \begin{bmatrix} A_1 \\ \boldsymbol{B_2} \end{bmatrix} \begin{matrix} = \\ \neq \end{matrix} \begin{bmatrix} \boldsymbol{B_1} \\ \boldsymbol{B_2} \end{bmatrix} \begin{matrix} \neq \\ = \end{matrix} \begin{bmatrix} \boldsymbol{B_1} \\ A_2 \end{bmatrix}?$$

From a purely formal point of view there are no designated "outer" realms, no designated points in the structure and there is no reason to cancel one of the two possible solutions. Creating concrete forms of this structure depends on *decisions* which a user has to make. If a thinker is able to differentiate between at least two independent dimensions with respect to two apparently mutually exclusive realms, he establishes a way to *create* a third realm simply by *rearranging* the known features in two or more dimensions. The nice property is that the new realm is not an independent thirdness but it is a *composition* out of the given. Furthermore, the new realm has something in common in one dimension and is different in the other dimension with respect to both given realms. Intermediate cases allow to *bridge* different poles and lead to a specific understanding of *internal harmony*. Frege expresses this hope in the following way:

> It is possible to view the signs of arithmetic, geometry, and chemistry as realizations, for specific fields, of Leibniz's idea. The ideography proposed here adds a new one to these fields, indeed the central one, which borders on all the others. If we take our departure from there, we can with the greatest expectation of success proceed to fill the gaps in the existing formula languages, connect their hitherto separated fields into a

> single domain, and extend this domain to include fields that up to now have lacked such a language. (FREGE 1970, p. 7: Jena, 18 December 1878).[5]

To state the concept (2-dimensionally) *structured intermediate case* precisely we have to be aware of the above listed *frame conditions*. With respect to Wittgenstein's remark we can differentiate the following two types:

(1) If the relevant frame conditions are already *given*, we call the specification of *solutions* with respect to intermediate cases *finding* intermediate cases.

(2) If a thinker has to *create* the frame conditions by herself which *enables* her (makes it possible at all) to find solutions, we call this type *inventing* intermediate cases.

The second type can be highly theoretically motivated. Therefore, we have a theoretically oriented reading of intermediate cases, which is different from but not excluded by Wittgenstein's main use of the term. Wittgenstein's use incorporates the philosophically interesting cases where we are not able to give appropriate frame conditions, i.e., language games without a plausible chance to make these conditions (rules) explicit.

2. Kant's *synthetic judgments a priori* as two-dimensionally structured intermediate cases

There are a lot of famous *–ism-dichotomies* in philosophy:

- rationalism vs. empirism (empiricism),
- materialism vs. idealism,
- realism (platonism) vs. nominalism,
- monism vs. dualism,
- infinitism vs. finitism etc.

A lot of logical systems (classical logic, many modal logics etc.) assume *bivalence*, i.e., the dichotomy *true* (being true, the true, t, 1, $\top$) vs. *false* (being false, the false, f, 0, $\bot$). We can understand well-known philosophical concepts as mutually exclusive pairs:

- mind (soul) vs. body,
- analytic vs. synthetic,
- a priori vs. a posteriori
- discreteness vs. continuity etc.

[5] „Man kann in den arithmetischen, geometrischen, chemischen Zeichen Verwirklichungen des Leibnizischen Gedankens für einzelne Gebiete sehen. Die hier vorgeschlagene Begriffsschrift fügt diesen ein neues hinzu und zwar das in der Mitte gelegene, welches allen andern benachbart ist. Von hier aus lässt sich daher mit der grössten Aussicht auf Erfolg eine Ausfüllung der Lücken der bestehenden Formelsprachen, eine Verbindung ihrer bisher getrennten Gebiete zu dem Bereiche einer einzigen und eine Ausdehnung auf Gebiete ins Werk setzen, die bisher einer solchen ermangelten." (FREGE 1879, p. VI: Jena, 18. Dezember 1878).

We assume a *realm* R, which can be subdivided into exactly two *mutually exclusive and complementary sub-regions* A and B. R – consisting of A and B – can be represented by the following picture:

A	B

The relation between A and B can be characterized by using an appropriate negation *not*. Then the following pictures would express the same situation:

A	*not* A

or

not B	B

or

not B	*not* A

There are many, many philosophical and/or logical reasons resp. motivations to give up this dichotomy:

- preferring *monism* or *holism*,
- trying to find a separate *third –ism*,
- postulating a *third* (*truth*) *value* (*indeterminate, uncertain, vague*, i, $\frac{1}{2}$, etc.),
- accepting a *truth-value gap* (for representing category mistakes, vagueness) etc.

If we keep A and/or B as *atoms* without any inner logical form and the *linearity* of the picture (*1-dimensional point of view*), there is only one possibility to do that: We have to switch from a dichotomic to a *trichotomic* picture:

A	B	C

The order of A, B and C does not matter! E.g., there is no reason to put C *between* A and B.

A	C	B

To get C *between* A and B we seem to be forced to accept some *qualitative* (*logical*) *order* or a kind of *quantitative scale* like in Aristotle's *Mesotes*-approach.

- Looking at 3-valued logics you can have values "t" (true), "i" (indeterminate) and "f" (false) *without any scale* or you can use an arithmetic style of naming values: "1" (true), "$\frac{1}{2}$" (indeterminate) and "0" (false) with $1 > \frac{1}{2} > 0$ that gives you an order.

- If the realm is *fixed*, then A together with B has to be "smaller" than the complete realm. Usually we say then that A and B are *contraries*: *A and B* cannot be "true" together, but *A or B* does not give you the complete realm (*A and B* can be "false" together).

- Another possibility would be subdividing one of our A, B, respectively:

A	B_1	B_2

or

A_1	A_2	B

Kant and Frege were confronted with several well-known dichotomies:

Kant's dichotomy 1 with respect to *judgments*:

analytic [not synthetic]	synthetic [not analytic]

Kant's dichotomy 2 with respect to *judgments*:

a priori [not a posteriori]	a posteriori [not a priori]

Frege's dichotomy 1 with respect to *objects*:

things of the outer world "Dinge der Außenwelt" [not ideas]	Ideas "Vorstellungen" [not things of the outer world]

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Frege's dichotomy 2 with respect to *objects*:

can be percieved by the senses "können mit den Sinnen wahrgenommen werden" *objective* [not subjective]	cannot be percieved by the senses "können nicht mit den Sinnen wahrgenommen werden" *subjective* [not objective]

Kant was aware of the famous *dichotomy* between *truths of facts* ("Tatsachenwahrheiten") and *truths of reason* ("Vernunftwahrheiten"). Judgments which represent truths of facts seem to be *synthetic* if and only if they are *a posteriori*. Judgments which represent truths of reason seem to be *analytic* if and only if they are *a priori*. Then we get only 2 kinds of judgments:

analytic a priori	synthetic a posteriori

If we look at *the realm of sciences*, a possible picture can be

analytic a priori	synthetic a posteriori
formal sciences (mathematics, logic)	*empirical* sciences (physics)

But what is the place of *metaphysics as science* in this realm? How can we create a *third* area, a third kind of judgments? Is it possible to understand *metaphysics* as a *linking/connecting piece* in the form of a 2-dimensionally structured *intermediate case* connecting formal and empirical sciences without establishing a separate thirdness?

analytic a priori	???	synthetic a posteriori

Kant begins his *Prolegomena to Any Future Metaphysics* with the introduction of his core concept *synthetic judgments a priori*:

> Whether this distinguishing feature consists in a difference of the *object* or the *source of cognition*, or even of the *type of cognition*, or some if not all of these things together, the idea of the possible science and its territory depends first of all upon it." (KANT 2004, p. 15: § 1).[6]

This formulation is ingenious and programmatic in several respects: (a) Kant is searching for the "distinguishing feature" of knowledge. But we have to differentiate between several dimensions (criteria): (D1) The feature can consist in "a difference of the object" (Erkenntnisobjekt) – later explained as the difference between *analytic* (A_1) and *synthetic* (B_1). (D2) It can consist in a difference of "the source" (Erkenntnisart) – later explicated as the difference between *a priori* (A_2) and *a posteriori* (B_2). (D3) But it can also consist in a difference of "the type of cognition". This could be C_1 and C_2. Kant states clearly that it is possible to characterize knowledge with respect to *disparate* dimensions. (b) The fundamental idea is expressed by "or some if not all of these things together"! Kant decided to take into consideration only two respects/dimensions: (D1) difference of the object and (D2) difference of the source and to put these two things *together*! But how?

Kant characterizes D1 consisting of A_1 *analytic* [not synthetic] and B_1 *synthetic* [not analytic] as follows:

> Analytic judgments [A_1][7] say nothing in the predicate except what was actually thought already in the concept of the subject, though not so clearly nor with the same consciousness. If I say: All bodies are extended, then I have not in the least amplified my concept of body, but have merely resolved it, since extension, although not explicitly said of the former concept prior to the judgment, nevertheless was actually thought of it; the judgment is therefore analytic. By contrast, the proposition: Some bodies are heavy, contains something in the predicate that is not actually thought in the general concept of body; it therefore augments my cognition, since

[6] „Dieses Eigentümliche mag nun in dem Unterschiede des Objekts, oder der Erkenntnisquellen, oder auch der Erkenntnisart, oder einiger, wo nicht aller dieser Stücke zusammen, bestehen, so beruht darauf zuerst die Idee der möglichen Wissenschaft und ihres Territorium." (KANT 1913, p. 13: §1).

[7] All additions in square brackets within quotations throughout the whole text are mine.

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it adds something to my concept, and must therefore be called a synthetic judgment [B_1]. (KANT 2004, p. 16: § 2a)[8]

Each judgment is *analytic* or *synthetic* but not both. I.e., *analytic* and *synthetic* represent a true dichotomy in D1 represented by

Judg	ments
analytic	synthetic

Kant characterizes D2 consisting of A_2 *a priori* [not a posteriori] and B_2 *a posteriori* [not a priori] as follows:

> Therefore it will be based upon neither outer experience, which constitutes the source of physics proper, nor inner, which provides the foundation of empirical psychology. It is therefore cognition a priori [A_2], or from pure understanding and pure reason. (KANT 2004, p. 15: § 1). [9]

> There are synthetic [B_1] judgments *a posteriori* B_2 whose origin [= source! I.M.] is empirical. (KANT 2004, p. 16: § 2a).[10]

Each judgment is *a priori* A_2 or *a posteriori* B_2 but not both. I.e., a priori and a posteriori represent a true dichotomy in D2 illustrated by

Judg	ments
a priori	a posteriori

[8] „Analytische Urteile [A_1] sagen im Prädikate nichts, als das, was im Begriffe des Subjekts schon wirklich, obgleich nicht so klar und mit gleichem Bewußtsein gedacht war. Wenn ich sage: alle Körper sind ausgedehnt, so habe ich meinen Begriff vom Körper nicht im mindesten erweitert, sondern ihn nur aufgelöset, indem die Ausdehnung von jenem Begriffe schon vor dem Urteile, obgleich nicht ausdrücklich gesagt, dennoch wirklich gedacht war; das Urteil ist also analytisch. Dagegen enthält der Satz: einige Körper sind schwer, etwas im Prädikate, was in dem allgemeinen Begriffe vom Körper nicht wirklich gedacht wird, er vergrößert also meine Erkenntnis, indem er zu meinem Begriffe etwas hinzutut, und muß daher ein synthetisches Urteil [B_1] heißen." (KANT 1913, p. 14: § 2a).

[9] „Also wird weder äußere Erfahrung, welche die Quelle der eigentlichen Physik, noch innere, welche die GrundJage der empirischen Psychologie ausmacht, bei ihr zum Grunde liegen. Sie ist also Erkenntnis a priori, oder aus reinem Verstande und reiner Vernunft." (KANT 1913, p. 13 f.: §1).

[10] „Es gibt synthetische Urteile a posteriori, deren Ursprung empirisch ist . . . " (KANT 1920, p. 15: § 2c).

Kant's great idea was giving up the identification of analytic with a priori as well as the identification of synthetic with a posteriori!

Judg	ments
analytic	synthetic
a priori	a posteriori

Judgments can be characterized in two distinct dimensions which he calls "objects" (analytic vs. synthetic) and "source of cognition" (a priori vs. a posteriori). He is *not* looking for a trichotomy, a third kind of judgments which is totally distinct from analytic/a priori judgments and synthetic/a posteriori judgments. His approach *keeps* both dichotomies. Everything which is needed is already given, but we have to rearrange it in a new manner. We have to give up the presupposition that each (dichotomic, trichotomic, *n*-tomic) distinction must have a linear order, a 1-dimensional structure. We can switch to 2-*dimensional* (possibly *higher-dimensional*) forms.

A third concept can be a rearrangement/a composition out of two 2-dimensional concepts in such a way that (a) it has something in common with both given concepts but with respect to different dimensions and (b) it is concurrently distinct from both given concepts in the remaining dimensions of the two given concepts.

Let A_1 and B_1 be the characterization of two concepts A and B in their first dimension, respectively. Let A_2 and B_2 the characterization of these two concepts A and B in their second dimension, respectively. This already opens the possibility of composing two different third concepts as 2-dimensionally structured intermediate cases. In Kant's case $\begin{bmatrix} B_1 \\ A_2 \end{bmatrix}$ is the form of *synthetic judgments a priori* (option 1). $\begin{bmatrix} A_1 \\ B_2 \end{bmatrix}$ represents *analytic judgments a posteriori* (option 2). The two options have nothing in common.

$$1) \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \begin{matrix} \neq \\ = \end{matrix} \begin{bmatrix} B_1 \\ A_2 \end{bmatrix} \begin{matrix} = \\ \neq \end{matrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} : \begin{bmatrix} \text{analytic} \\ \text{a priori} \end{bmatrix} \begin{matrix} \neq \\ = \end{matrix} \begin{bmatrix} \text{synthetic} \\ \text{a priori} \end{bmatrix} \begin{matrix} = \\ \neq \end{matrix} \begin{bmatrix} \text{synthetic} \\ \text{a posteriori} \end{bmatrix}$$

$\begin{bmatrix} B_1 \\ A_2 \end{bmatrix}$ is created by Kant.

$$2)\ \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \begin{matrix} = \\ \neq \end{matrix} \begin{bmatrix} A_1 \\ B_2 \end{bmatrix} \begin{matrix} \neq \\ = \end{matrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} : \begin{bmatrix} \text{analytic} \\ \text{a priori} \end{bmatrix} \begin{matrix} = \\ \neq \end{matrix} \begin{bmatrix} \text{analytic} \\ \text{a posteriori} \end{bmatrix} \begin{matrix} \neq \\ = \end{matrix} \begin{bmatrix} \text{synthetic} \\ \text{a posteriori} \end{bmatrix}$$

$\begin{bmatrix} A_1 \\ B_2 \end{bmatrix}$ is not accepted by Kant.

Kant is very clear about choosing option (1) and not allowing option (2)[11] with the following consequences:

1a) "All analytic judgments rest entirely on the principle of contradiction and are by their nature *a priori* cognitions, whether the concepts that serve for their material be empirical or not." (KANT 2004, p. 17: § 2b).[12]

I.e., for each judgment holds: If this judgment is *analytic*, then it is *a priori*. The converse does not hold. In this respect Frege follows Kant (cf. footnote 17)!

1b) "Judgments of experience [a posteriori, I.M.], as such, are all synthetic." (KANT 1998, p. 142: IV, B11).[13]

I.e., for each judgment holds: If this judgment is *a posteriori*, then it is *synthetic*. The converse does not hold either. This means: Naming a judgment "analytic" in dimension 1 is sufficient to get "analytic judgment a priori". Naming a judgment "a posteriori" in dimension 2 is sufficient for "synthetic judgment a posteriori". But there is no shorter name for "synthetic judgment a priori"!

3. Frege's *thoughts* as two-dimensional intermediate cases in the style of Kant

If we look at the relation between Kant and Frege regarding *equations as judgments/sentences in arithmetic*, we get opposite answers. Kant says: "*Mathematical judgments* are one and all synthetic." (KANT 2004, p. 18: § 2c2).[14] Against this Frege says: "I hope I may claim in the present work to have made it probable that the laws of arithmetic are analytic judgements and consequently a priori." (FREGE 1953, p. 99: V. Conclusion, § 87).[15] But Frege has a similar

[11] It is worth mentioning that KRIPKE in Naming and Necessity (1972) uses a similar pattern and fills this "gap" by postulating necessary (taken for analytical) a posteriori truths. He uses both types of 2-dimensional intermediate cases.

[12] "Alle analytischen Urteile beruhen gänzlich auf dem Satze des Widerspruchs, und sind ihrer Natur nach Erkenntnisse *a priori*, die Begriffe, die ihnen zur Materie dienen, mögen empirisch sein, oder nicht." (KANT 1920, p. 15: § 2b).

[13] "Erfahrungsurteile, als solche, sind insgesamt synthetisch." (KANT 1998, p. 59: Einleitung IV, B11).

[14] „Mathematische Urteile sind insgesamt synthetisch." (KANT 2004, p. 16: § 2c2).

[15] "Ich hoffe in dieser Schrift wahrscheinlich gemacht zu haben, daß die arithmetischen Gesetze analytische Urteile und folglich a priori sind." (FREGE 1884, p. 99: V. Schluss, § 87).

project like Kant: How is logic as science possible? What are the objects of logic? Like in the Kantian case there seems to be no room for a third realm.

> A person who is still untouched by philosophy knows first of all things which he can see and touch, in short, perceive with the senses, such as trees, stones and houses, and he is convinced that another person equally can see and touch the same tree and the same stone which he himself sees and touches. Obviously no thought belongs to these things. Now can he, nevertheless, stand in the same relation to a person as to a tree? ... Even an unphilosophical person soon finds it necessary to recognize an inner world distinct from the outer world, a world of sense-impressions, of creations of his imagination, of sensations, of feelings and moods, a world of inclinations, wishes and decisions. For brevity I want to collect all these, with the exception of decisions, under the word "idea". / Now do thoughts belong to this inner world? Are they ideas? They are obviously not decisions. (FREGE 1956, p. 298 f.).[16]

Frege's dichotomy with respect to *objects*

A : things of the outer world	B : ideas (everything of the inner world without decisions)
[*not* ideas]	[not things of the outer world]

In the next step Frege introduces his dimensions. D1a: "First: Ideas cannot be seen or touched, cannot be smelled, nor tasted, nor heard."[17] (FREGE 1956, p. 299). D1b "ideas are had"[18] (ibid.)

[16] "Der von der Philosophie noch unberührte Mensch kennt zunächst Dinge, die er sehen, tasten, kurz, mit den Sinnen wahrnehmen kann, wie Bäume, Steine, Häuser, und er ist überzeugt, daß ein anderer denselben Baum, denselben Stein, den er selbst sieht und tastet, gleichfalls sehn und tasten kann. Zu diesen Dingen gehört ein Gedanke offenbar nicht. Kann er nun trotzdem den Menschen als derselbe gegenüberstehn wie ein Baum ... ? Auch der unphilosophische Mensch sieht sich bald genötigt, eine von der Außenwelt verschiedene Innenwelt anzuerkennen, eine Welt der Sinneseindrücke, der Schöpfungen seiner Einbildungskraft, der Empfindungen, der Gefühle und Stimmungen, eine Welt der Neigungen, Wünsche und Entschlüsse. Um einen kurzen Ausdruck zu haben, will ich dies mit Ausnahme der Entschlüsse unter dem Worte ‚Vorstellung' zusammenfassen. / Gehören nun die Gedanken dieser Innenwelt an? Sind sie Vorstellungen? Entschlüsse sind sie offenbar nicht." (FREGE 1918/19a, p. 66.)

[17] „Vorstellungen können nicht gesehen oder getastet, weder gerochen, noch geschmeckt, noch gehört werden." (FREGE 1918/19a, p. 67).

[18] „Vorstellungen werden gehabt." (ibid).

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things of the outer world A_1 $A_1 = \neg B_1$: can be percieved by the senses (We *see*, *touch*, *smell* etc. things.)	ideas B_1 $B_1 = \neg A_1$: can*not* be percieved by the senses (We *have* ideas.)

D2a: "ideas need a bearer. Things of the outer world are however independent"[19] (ibid.). D2b "every idea has only one bearer; no two men have the same idea"[20] (ibid., p. 300). D2a and D2b gives us *objectivity* and *invariance* of things of the outer world as well as the opposite for ideas: their *subjectivity* and *variance*:

things of the outer world A_2 $A_2 = \neg B_2$: objective / invariant [non-subjective & non variant]	ideas B_2 $B_2 = \neg A_2$:subjective / variant [non-objective & non invariant]

Neither things of the outer world nor ideas can be the logical objects wanted by Frege. Maybe we are inclined to say that *logical* objects are *abstract* objects!? But what does that mean? Frege's answer is:

> So the result seems to be: *thoughts* [C] are neither things of the outer world [A] nor ideas [B].
>
> A *third realm* [„set" of 2-dimensional intermediate cases C] must be recognized. *What belongs to this* [any C as element], corresponds with ideas [B], in that *it* [any C as element] cannot be perceived by the senses [$B_1 = \neg A_1$], but with *things* [A], in that *it* [any C as element] needs no bearer to the contents of whose consciousness to belong [$A_2 = \neg B_2$]. (ibid., p 302)[21]

The first dimension D1 is *the perceivableness-dimension*: perceivable by senses [A_1] vs. *not* perceivable by senses [B_1]. The second dimension D2

[19] „Vorstellungen bedürfen eines Trägers. Die Dinge der Außenwelt sind im Vergleiche damit selbständig." (ibid).

[20] „Jede Vorstellung hat nur einen Träger; nicht zwei Menschen haben dieselbe Vorstellung." (ibid, p. 68).

[21] „So scheint das Ergebnis zu sein: Die Gedanken [C] sind weder Dinge der Außenwelt [A], noch Vorstellungen [B]. / Ein drittes Reich [Gesamtheit der C] muß anerkannt werden. Was zu diesem gehört, stimmt mit den Vorstellungen [B] darin überein, daß es, nicht mit den Sinnen wahrgenommen werden kann [$B_1 = \neg A_1$], mit den Dingen [A] aber darin, daß es keines Trägers bedarf, zu dessen Bewußtseinsinhalte es gehört [$A_2 = \neg B_2$]." (FREGE 1918/19a, p.69).

is *the bearer-dimension*: objective (needs no bearer, invariant) [A_2] vs. subjective (needs a bearer) [B_2]. A thing (of the outer world) A is perceivable by senses in D1 and objective in D2. A thing A can be 2-dimensionally represented as $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$. An idea B is the absolute opposite: not perceivable by senses in D1 and subjective in D2. An idea B can be 2-dimensionally represented as $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$. Frege seems to assume as a further frame condition *bivalence for both dimensions*:

a) Everything is exactly one of A_1 perceivable by senses / B_1 not perceivable by senses (D1).

b) Everything is exactly one of A_2 objective (not-subjective) / B_2 not-objective (subjective) (D2).

This gives us $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ and $B = \begin{bmatrix} \textit{not } A_1 \\ \textit{not } A_2 \end{bmatrix}$. We will come back to this point in the next section.

By establishing all these conditions Frege invented („created“ the room for finding) an appropriate 2-dimensional intermediate case to answer the question regarding the general possibility of logic as a formal theory, i.e., the question “What are the objects of logic?” From a purely structural point of view there are two candidates for C: $\begin{bmatrix} A_1 \\ B_2 \end{bmatrix}$ and $\begin{bmatrix} B_1 \\ A_2 \end{bmatrix}$. $\begin{bmatrix} A_1 \\ B_2 \end{bmatrix}$ yields $\begin{bmatrix} \text{perceivable by senses} \\ \text{subjective} \end{bmatrix}$. There is no hint that Frege considered this case seriously. Maybe *introspection* could be a case!?

$\begin{bmatrix} B_1 \\ A_2 \end{bmatrix}$ represents $\begin{bmatrix} \text{not perceivable by senses} \\ \text{objective} \end{bmatrix}$. That is exactly the form of Frege’s *thoughts*. Each Element of Frege’s third realm has this form and is called *the thought*. *Each thought* is a 2-dimensionally structured intermediate case of the form $\begin{bmatrix} B_1 \\ A_2 \end{bmatrix}$ between things of the outer world of the form $\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ and ideas of the form $\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$. Instead of a 1-dimensional reading things A – thoughts C – ideas B we get $\begin{bmatrix} A_1 \\ \boldsymbol{A_2} \end{bmatrix}$ – $\begin{bmatrix} \boldsymbol{B_1} \\ \boldsymbol{A_2} \end{bmatrix}$ – $\begin{bmatrix} \boldsymbol{B_1} \\ B_2 \end{bmatrix}$ or

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$$\begin{bmatrix}\text{perceivable by senses}\\ \text{objective}\end{bmatrix}\begin{matrix}\neq\\=\end{matrix}\begin{bmatrix}\text{not perceivable by senses}\\ \text{objective}\end{bmatrix}\begin{matrix}=\\ \neq\end{matrix}\begin{bmatrix}\text{not perceivable by senses}\\ \text{subjective}\end{bmatrix}$$

" $\begin{matrix}=\\ \neq\end{matrix}$ " indicates the 2-dimensional relation between *thoughts* and *ideas* ($C - B$) what Frege calls "corresponds with" and " $\begin{matrix}\neq\\ =\end{matrix}$ " the 2-dimensional relation between *thoughts* and *things* ($C - A$) expressed in English by "but with". That is not a good translation. In the German version it is very clear that Frege speaks about identity ("stimmt überein mit"). Be careful! In the first case we have the identity in the first dimension ("stimmt mit den Vorstellungen [B] darin überein, daß *es* nicht mit den Sinnen wahrgenommen werden kann": B_1-identity) and in the second case the identity in the second dimension ("mit den Dingen [A] aber darin, daß es keines Trägers bedarf, zu dessen Bewußtseinsinhalte es gehört": A_2-identity).

4. Frege's *thoughts* and their negation reconstructed in a 2-dimensional logic

With respect to the *bivalence principle* the schemes

$$\begin{bmatrix}A_1\\A_2\end{bmatrix}\begin{matrix}=\\ \neq\end{matrix}\begin{bmatrix}A_1\\B_2\end{bmatrix}\begin{matrix}\neq\\ =\end{matrix}\begin{bmatrix}B_1\\B_2\end{bmatrix}\text{and}\begin{bmatrix}A_1\\A_2\end{bmatrix}\begin{matrix}\neq\\ =\end{matrix}\begin{bmatrix}B_1\\A_2\end{bmatrix}\begin{matrix}=\\ \neq\end{matrix}\begin{bmatrix}B_1\\B_2\end{bmatrix}$$

could be further reduced if we claim – index-independently – that $B_1 = \neg A_1$ and $B_2 = \neg A_2$:

$$\begin{bmatrix}A_1\\A_2\end{bmatrix}\begin{matrix}=\\ \neq\end{matrix}\begin{bmatrix}A_1\\ \neg A_2\end{bmatrix}\begin{matrix}\neq\\ =\end{matrix}\begin{bmatrix}\neg A_1\\ \neg A_2\end{bmatrix}\text{and}\begin{bmatrix}A_1\\A_2\end{bmatrix}\begin{matrix}\neq\\ =\end{matrix}\begin{bmatrix}\neg A_1\\A_2\end{bmatrix}\begin{matrix}=\\ \neq\end{matrix}\begin{bmatrix}\neg A_1\\ \neg A_2\end{bmatrix}$$

Please note that the use of our (classical) negation symbol "¬" inside square brackets is part of our language of analysis. It is used inside 2- dimensional constructions which – as a whole – represent possible reduced forms of 2-dimensionally structured intermediate cases. If we want to explicate the relation between $\begin{bmatrix}A_1\\ \neg A_2\end{bmatrix}$ and $\begin{bmatrix}\neg A_1\\ A_2\end{bmatrix}$, we can try to characterize a *reduction operator* # which is a syntactically unary connective but takes 2-dimensional arguments of the form $\begin{bmatrix}A_1\\ \neg A_2\end{bmatrix}$ as input and yields $\begin{bmatrix}\neg A_1\\ A_2\end{bmatrix}$ output and vice versa:

$$\#\begin{bmatrix} A_1 \\ \neg A_2 \end{bmatrix} = \begin{bmatrix} \neg A_1 \\ A_2 \end{bmatrix} \text{ and } \#\begin{bmatrix} \neg A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} A_1 \\ \neg A_2 \end{bmatrix}.$$

We can characterize our negation # by the reduction rule R# applicable to any 2-dimensional construction of the form $\begin{bmatrix} A \\ B \end{bmatrix}$:

$$\#\begin{bmatrix} A \\ B \end{bmatrix} \Longrightarrow \begin{bmatrix} \neg A \\ \neg B \end{bmatrix}.$$

The application of this reduction rule runs as follows: Let X be any formula in which an expression of the form $\#\begin{bmatrix} A \\ B \end{bmatrix}$ has one or more occurrences. Then R# allows to replace any such occurrence of $\#\begin{bmatrix} A \\ B \end{bmatrix}$ by $\begin{bmatrix} \neg A \\ \neg B \end{bmatrix}$. Take the following example:

1.	$\#\#\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix}$	
2.	$\#\begin{bmatrix} \neg A \\ \neg B \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix}$	R#
3.	$\begin{bmatrix} \neg\neg A \\ \neg\neg B \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix}$	R#
4.	$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix}$ because of $\models \neg\neg A \equiv A$	

"$\models A$" says that A is *classically valid*. Inside square brackets we can use *the replacement of logically equivalent expressions* without any restriction. The rule R# is called a *reduction* rule. In general, reduction rules allow the stepwise elimination of all occurrences of any n-ary reduction operator outside square brackets completely and reduces any more complex structure finally to a structure of the form $\begin{bmatrix} A \\ B \end{bmatrix}$. Here is the general form of reduction rules of unary connectives applied to 2-dimensional inputs "producing" 2-dimensional outputs:

$$*\begin{bmatrix} A \\ B \end{bmatrix} \Longrightarrow \begin{bmatrix} \varphi^2 AB \\ \psi^2 AB \end{bmatrix},$$

where φ^2 and ψ^2 are – not necessarily different – classical binary connectives. It is, by definition. that $\neg A$ and $\neg B$ in our reduction rule R# are expressible by an appropriate choice of φ^2 (ψ^2).

If we add the identification of both dimensions ($A_1 = A_2$), we get

$$\begin{bmatrix} A_{(1)} \\ A_{(1)} \end{bmatrix} \begin{matrix} = \\ \neq \end{matrix} \begin{bmatrix} A_{(1)} \\ \neg A_{(1)} \end{bmatrix} \begin{matrix} \neq \\ = \end{matrix} \begin{bmatrix} \neg A_{(1)} \\ \neg A_{(1)} \end{bmatrix} \text{and} \begin{bmatrix} A_{(1)} \\ A_{(1)} \end{bmatrix} \begin{matrix} \neq \\ = \end{matrix} \begin{bmatrix} \neg A_{(1)} \\ A_{(1)} \end{bmatrix} \begin{matrix} = \\ \neq \end{matrix} \begin{bmatrix} \neg A_{(1)} \\ \neg A_{(1)} \end{bmatrix}.$$

If we further assume $\begin{bmatrix} A \\ A \end{bmatrix} = A$ and $\begin{bmatrix} \neg A \\ \neg A \end{bmatrix} = \neg A$, we get

$$A \begin{matrix} = \\ \neq \end{matrix} \begin{bmatrix} A \\ \neg A \end{bmatrix} \begin{matrix} \neq \\ = \end{matrix} \neg A \text{ and } \neg B \begin{matrix} \neq \\ = \end{matrix} \begin{bmatrix} B \\ \neg B \end{bmatrix} \begin{matrix} = \\ \neq \end{matrix} B.$$

Finally, if we substitute, e.g., "$\top$" (*the true*) for "A" and "$\bot$" (*the false*) for "$\neg A$" we get

$$\top \begin{matrix} = \\ \neq \end{matrix} \begin{bmatrix} \top \\ \bot \end{bmatrix} \begin{matrix} \neq \\ = \end{matrix} \bot \text{ and } \bot \begin{matrix} \neq \\ = \end{matrix} \begin{bmatrix} \top \\ \bot \end{bmatrix} \begin{matrix} = \\ \neq \end{matrix} \top.$$

Now we are able to characterize the difference between Frege's *philosophical understanding* and his *theoretical explication* of thoughts. The general form of thoughts in his *philosophical* context is only $\begin{bmatrix} B_1 \\ A_2 \end{bmatrix}$ which symbolizes $\begin{bmatrix} \text{not perceivable by senses} \\ \text{objective} \end{bmatrix}$. There is no hint that he considers the second possibility $\begin{bmatrix} A_1 \\ B_2 \end{bmatrix}$ which represents $\begin{bmatrix} \text{perceivable by senses} \\ \text{subjective} \end{bmatrix}$. Frege's context was to motivate his undefinable basic concept *the thought* in order to answer the *philosophical* question: "How is logic as (formal) science possible?". In this context thoughts are intermediate cases between things of the outer world and ideas. The opposite cases ("anti-thoughts") – instances of $\begin{bmatrix} \text{perceivable by senses} \\ \text{subjective} \end{bmatrix}$ (introspections) – are not relevant for Frege. But Frege uses thoughts as *logical* objects in his formalism as well. Now they have *two* possible forms with equal logical weight: $\begin{bmatrix} A \\ \neg A \end{bmatrix}$ and $\begin{bmatrix} \neg A \\ A \end{bmatrix}$. We can call both kinds of thoughts *classical thoughts*. $\begin{bmatrix} A \\ \neg A \end{bmatrix}$ and $\begin{bmatrix} \neg A \\ A \end{bmatrix}$ represent *opposites of each other*: $\begin{bmatrix} A \\ \neg A \end{bmatrix}$ is *the opposite thought with respect to* $\begin{bmatrix} \neg A \\ A \end{bmatrix}$ and $\begin{bmatrix} \neg A \\ A \end{bmatrix}$ is *the opposite thought with respect to* $\begin{bmatrix} A \\ \neg A \end{bmatrix}$. Furthermore, we have to differentiate between *empirical* thoughts like $\begin{bmatrix} p \\ \neg p \end{bmatrix}$ and their opposites like $\begin{bmatrix} \neg p \\ p \end{bmatrix}$ on the one hand and *logical* thoughts like $\begin{bmatrix} p \vee \neg p \\ p \wedge \neg p \end{bmatrix}$, $\begin{bmatrix} \top \\ \bot \end{bmatrix}$ and their opposites like $\begin{bmatrix} p \wedge \neg p \\ p \vee \neg p \end{bmatrix}$, $\begin{bmatrix} \bot \\ \top \end{bmatrix}$ on the other hand. The *Begriffsschrift* is Frege's proof system of *logical thoughts* which yields as a result of *judgments* – indicated by the vertical stroke – 1-dimensional classically valid formulas, formulas which are logically equivalent with $\top$, formulas which denote *the true*.

We can use the negation connective "#" to get the opposite thought out of a given thought. But in this case we need step 4 in our prove of $\#\#\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix}$ above. In our 2-dimensional reading of thoughts we have another choice: the negation connective $\updownarrow$ characterized by the reduction rule R$\updownarrow$.

$$\updownarrow \begin{bmatrix} A \\ B \end{bmatrix} \Rightarrow \begin{bmatrix} B \\ A \end{bmatrix}.$$

1.	$\updownarrow\updownarrow \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix}$	
2.	$\updownarrow \begin{bmatrix} B \\ A \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix}$	R$\updownarrow$
3.	$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix}$	R$\updownarrow$

It is clear the # and $\updownarrow$ give different results if applied to arbitrary 2-dimensional expressions: $\#\begin{bmatrix} A \\ B \end{bmatrix} \neq \updownarrow \begin{bmatrix} A \\ B \end{bmatrix}$ because of $\begin{bmatrix} \neg A \\ \neg B \end{bmatrix} \neq \begin{bmatrix} B \\ A \end{bmatrix}$. But if we apply both negation connectives exclusively to thoughts we get coincidence: $\#\begin{bmatrix} A \\ \neg A \end{bmatrix} = \updownarrow \begin{bmatrix} A \\ \neg A \end{bmatrix}$ because

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of $\begin{bmatrix} \neg A \\ \neg\neg A \end{bmatrix} = \begin{bmatrix} \neg A \\ A \end{bmatrix}$. We think that "$\updownarrow$" is an interesting way of looking at Frege's understanding of logical negation with respect to *opposite thoughts* in the context of our 2-dimensional language of analysis!

> Thus for every thought [$\begin{bmatrix} A \\ \neg A \end{bmatrix}$ or $\begin{bmatrix} \neg A \\ A \end{bmatrix}$, respectively] there is a contradictory [footnote: We could also say 'an opposite thought.'] thought [$\begin{bmatrix} \neg A \\ A \end{bmatrix}$ or $\begin{bmatrix} A \\ \neg A \end{bmatrix}$, respectively]; we acknowledge the falsity of a thought by admitting the truth of its contradictory[22]. The sentence that expresses the contradictory thought is formed from the expression of the original thought by means of a negative word [$\updownarrow$: $\updownarrow \begin{bmatrix} A \\ \neg A \end{bmatrix}$ or $\updownarrow \begin{bmatrix} \neg A \\ A \end{bmatrix}$, respectively]. (FREGE 1960, p. 131).[23]

Negation "$\updownarrow$" is a reduction connective (representing the "negative word") which takes an arbitrary thought and yields the opposite thought simply by inverting the order within special 2-dimensional expressions.

In MAX 2003 I showed that it is possible to establish a 2-dimensional logic L^2 which uses the negation connective "$\updownarrow$" and yields as its theorems/tautologies expressions of the form $\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ with $\models A_1$ ($\vdash A_1$) together with $?A_2$ ($\dashv A_2$):[24]

$$\models_{10} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} =_{df} \models A_1 \;\&?A_2 (\vdash_{10} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} =_{df} \vdash A_1 \;\&\; \dashv A_2).$$

[22] Let me mention that the translation of "widersprechender" by "contradictory" has possibly the misleading connotation that the formal explication of "the negative word" can only be reached by "$\neg$" *in the* $1-$*dimensional and* " #" *in the* $2-$*dimensiomal reading*.

[23] "Zu jedem Gedanken [$\begin{bmatrix} A \\ \neg A \end{bmatrix}$ bzw. $\begin{bmatrix} \neg A \\ A \end{bmatrix}$] gehört demnach ein ihm widersprechender [Fußnote: Man könnte auch sagen "ein entgegengesetzter".] Gedanke [$\begin{bmatrix} \neg A \\ A \end{bmatrix}$] bzw. $\begin{bmatrix} A \\ \neg A \end{bmatrix}$] derart, daß ein Gedanke dadurch als falsch erklärt wird, daß der ihm widersprechende als wahr anerkannt wird. Der den widersprechenden Gedanken ausdrückende Satz wird mittels eines Verneinungswortes [$\updownarrow$] aus dem Ausdrucke des ursprünglichen Gedankens gebildet [$\updownarrow \begin{bmatrix} A \\ \neg A \end{bmatrix}$ *bzw.* $\updownarrow \begin{bmatrix} \neg A \\ A \end{bmatrix}$]." (FREGE 1918/19b, p. 154).

[24] "$\Dashv$ A" says that "A" is a classical contradiction, $\vdash$ A that "A" is a classical theorem and $\dashv$ A that "*A*" is a classical anti-theorem.

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Let X be an arbitrary formula of L^2 containing exclusively elementary expressions of the form $\begin{bmatrix} p_i \\ \neg p_i \end{bmatrix}$. After a complete reduction of all the connectives occurring in X we get a formula of the form $\begin{bmatrix} A \\ \neg A \end{bmatrix}$ or of the form $\begin{bmatrix} \neg A \\ A \end{bmatrix}$:

$$\models_{10} \begin{bmatrix} A \\ \neg A \end{bmatrix} \text{ iff} \models A \text{ iff } ?\neg A. \models_{10} \begin{bmatrix} \neg A \\ A \end{bmatrix} \text{ iff} \models \neg A \text{ iff } ?A$$

Here is the general form of reduction rules of arbitrary binary connectives $\circledast$ applied to 2-dimensional inputs "producing" 2-dimensional outputs:

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \circledast \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \Longrightarrow \begin{bmatrix} \varphi^4 A_1 A_2 B_1 B_2 \\ \psi^4 A_1 A_2 B_1 B_2 \end{bmatrix}.$$

It is possible to embed the language of our familiar classical propositional logic L^1 into the language of this 2-dimensional logic L^2.[25] We call this sublanguage L^2_1. The basic ideas are:

(1) Each propositional variable p_i of L^1 is translated into an ordered pair $\begin{bmatrix} p_i \\ \neg p_i \end{bmatrix}$ in L^2.[26] $\begin{bmatrix} p_i \\ \neg p_i \end{bmatrix}$ is the syntactic form of *empirical elementary thoughts*. An alternative would be to put this "2-dimensionality" into expressions of the form $\begin{matrix} p_i \\ t \\ f \end{matrix}$ or $\begin{matrix} p_i \\ 1 \\ 0 \end{matrix}$, where "$p_i$" is an atomic sign of syntax and "t" and "f" ("1" and "0") are atomic semantic signs.

(2) The translation rules and the rules characterizing reduction connectives corresponding to classical junctors (negation "$\neg$", conjunction "$\wedge$", disjunction "$\vee$" etc.) are given in such a way that the complete reduction of each expression of the sublanguage of L^2_1 corresponding to the language of L^1 has finally the form $\begin{bmatrix} A \\ \neg A \end{bmatrix}$ or $\begin{bmatrix} \neg A \\ A \end{bmatrix}$, where "$A$" and "$\neg A$" represent well-formed formulas of classical propositional logic. All the entries within square brackets are 1-dimensional classical ones. But the occurrences of clas-

[25] Cf. MAX 2003.

[26] It does not matter whether we translate p of L[1] *into* $\begin{bmatrix} p \\ \neg p \end{bmatrix}$ or into $\begin{bmatrix} \neg p \\ p \end{bmatrix}$ in L^2 if we readjust the environment properly.

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sical operators ¬, ∧, ∨, ⊃ and ≡ within square brackets belong to our language of analysis. In this context they represent neither meaning nor sense if considered in isolation. Classical negation ¬ is not identical with our negation $\updownarrow$ in front of square brackets. $\neg\begin{bmatrix} A \\ B \end{bmatrix}$ and $\updownarrow A$ are not well-formed expressions.

(3) The sense of each tautology $\models A$ or theorem $\vdash A$ of L^1 is $\begin{bmatrix} A \\ \neg A \end{bmatrix}$ with $\models A \equiv \top$ and $\models \neg A \equiv \perp$ whereby $\top$ is a propositional constant naming the true and $\perp$ a propositional constant naming the false. The thought of any tautology can be expressed by $\begin{bmatrix} \top \\ \perp \end{bmatrix}$ and the thought of any contradiction can be expressed by $\begin{bmatrix} \perp \\ \top \end{bmatrix}$. Tautologies and contradictions are opposite thoughts: $\updownarrow\begin{bmatrix} \top \\ \perp \end{bmatrix} = \begin{bmatrix} \perp \\ \top \end{bmatrix}$ and $\updownarrow\begin{bmatrix} \perp \\ \top \end{bmatrix} = \begin{bmatrix} \top \\ \perp \end{bmatrix}$. A 2-dimensional tautology is a 2-dimensional expression – maybe the final result of applying reduction rules – with a classical tautology in the first and a classical contradiction in the second dimension. A 2-dimensional contradiction is a 2-dimensional expression with a classical contradiction in the first and a classical tautology in the second dimension.

Here is the complete list of translations:

L^1		L^2	R
p_i	$\Leftrightarrow$	$\begin{bmatrix} p_i \\ \neg p_i \end{bmatrix}$	
$\neg$	$\Leftrightarrow$	$\updownarrow$ with $\updownarrow\begin{bmatrix} A \\ B \end{bmatrix} \Rightarrow \begin{bmatrix} B \\ A \end{bmatrix}$	R$\updownarrow$
$\wedge$	$\Leftrightarrow$	$\sqcap$ with $\begin{bmatrix} A \\ B \end{bmatrix} \sqcap \begin{bmatrix} C \\ D \end{bmatrix} \Rightarrow \begin{bmatrix} A \wedge C \\ B \vee D \end{bmatrix}$	R$\sqcap$
$\vee$	$\Leftrightarrow$	$\sqcup$ with $\begin{bmatrix} A \\ B \end{bmatrix} \sqcup \begin{bmatrix} C \\ D \end{bmatrix} \Rightarrow \begin{bmatrix} A \vee C \\ B \wedge D \end{bmatrix}$	R$\sqcup$
$\supset$	$\Leftrightarrow$	$\rightarrow$ with $\begin{bmatrix} A \\ B \end{bmatrix} \rightarrow \begin{bmatrix} C \\ D \end{bmatrix} \Rightarrow \begin{bmatrix} A \supset C \\ \neg(D \supset B) \end{bmatrix}$	R$\rightarrow$

An example:

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L^1		L^2	R
$p \vee \neg p$	$\Leftrightarrow$	$\begin{bmatrix} p \\ \neg p \end{bmatrix} \sqcup \updownarrow \begin{bmatrix} p \\ \neg p \end{bmatrix}$	
		$\begin{bmatrix} p \\ \neg p \end{bmatrix} \sqcup \begin{bmatrix} \neg p \\ p \end{bmatrix}$	R$\updownarrow$
		$\begin{bmatrix} p \vee \neg p \\ \neg p \wedge p \end{bmatrix}$	R$\sqcup$
$\vDash p \vee \neg p$		$\vDash_{10} \begin{bmatrix} p \vee \neg p \\ \neg p \wedge p \end{bmatrix}$ because of $\vDash p \vee \neg p$ and $\Dashv \neg p \wedge p$	

5. Frege's negation and his trouble with Kant's negative judgments

We already know that in the sublanguage L_1^2 really any expression – after the complete application of reduction rules – reduces to one of these two forms: $\begin{bmatrix} A \\ \neg A \end{bmatrix}$ or $\begin{bmatrix} \neg A \\ A \end{bmatrix}$. Our negation "$\updownarrow$" transforms each thought into its opposite thought and vice versa. So far any thought is represented by a 2-*dimensional structure*. We can associate meanings[27] (the true/the false) with 1-dimensional structures simply. But in the sublanguage of L^2 which corresponds to the language of L^1 there is no possibility to produce expressions of the forms A or $\neg A$ which would be our candidates to represent meaning. We can declare that the logical form of the truth of "A" is simply A and that the logical form of the falsity of "A" is then $\neg A$. Of course, we could do that the other way around: A represents the falsity of "A" and $\neg A$ the truth of "A". In general, 2-dimensional expressions of L^2 represent thoughts and 1-dimensional expressions of L^2 represent meaning. What we need is the logical form of a connective which takes 2-dimenstional structures (thoughts) as inputs and yields 1-dimensional structures as output.

Our reduction operator "$\updownarrow$" applied to 2-dimensional arguments is – in our reconstruction – the syntactic counterpart of Frege's logical negation which GREIMANN (2018, p. 409-411, 413,

[27] I use "meaning" as the translation of Frege's "Bedeutung" despite the fact that it is very common to translate "Bedeutung" as "reference".

[28] E.g., „Semantic negation consists in the application of the logical function denoted by 'it is false that p' to a thought ..." (Greimann 2018, p. 410).

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417 f., 424) calls with respect to FREGE 1918/19b "semantic negation".[28] There seems to be another logical negation, called "pragmatic negation": "pragmatic negation in the act of asserting or judging a thought as false". (GREIMANN 2018, p. 410). Greimann uses a language of analysis which is different from ours. Relevant parts of his interpretation language are the use of "act" in the context of judging and the distinction between semantics and pragmatics. The main difference to our language of analysis is that we are interested in the explication of possible logical forms of thoughts, truth-values, negation, judgment etc. These forms are presented in a syntactic fashion.

> Are there two different ways of judging, of which one is used for the affirmative, and the other for the negative, answer to a question? Or is judging the same act in both cases? Does negating go along with judging? Or is negation part of the thought that underlies the act of judging? (FREGE 1960, p. 129).[29]

We can associate Greimann's way of speaking about judging the false – *pragmatic negation* – with Kant's way of speaking: *negative judgment*. The methodological problem behind the discussion is the alleged symmetry between negative and positive judgments and the relation between negating and negatively judging. From our point of view we have already stated that to negate a thought amounts to postulating its opposite thought. To negate a 2-dimensional argument by $\updownarrow$ gives us another 2-dimensional expression. But judging in general has not to be understood as an *act* but formally as *a connective that reduces a 2-dimensional thought to a 1-dimensional meaning*. Whether this meaning is the true or the false does not depend on this connective but on the formal structure of the argument. In this sense the speech of "pragmatic negation" is misleading. Here negativity is associated with negation and not with the inner structure of elementary classical thoughts. The effect of "pragmatic negation" does not come from the outside. It comes from the inside. If we look at $\updownarrow \begin{bmatrix} A \\ \neg A \end{bmatrix}$, we can observe that the reduction rule "$\updownarrow$" of R$\updownarrow$ is purely positive. "$\updownarrow$" "acts" like classical negation if all of its arguments are classical thoughts.

We interpret *judging* like negation as something which is neither positive nor negative as such. But the difference with negation is that judging re-

[29] „Gibt es zwei verschiedene Weisen des Urteilens, von denen jene bei der bejahenden, diese bei der verneinenden Antwort auf eine Frage gebraucht wird? Oder ist das Urteilen in beiden Fällen dasselbe? Gehört das Verneinen zum Urteilen? Oder ist die Verneinung Teil des Gedankens, der dem Urteilen unterliegt¿' (FREGE 1918/19b, p. 153).

duces dimensionality! It takes a 2-dimensional thought and yields a 1-dimensional meaning. Because Frege uses "|" (the *vertical* stroke or the *judgment* stroke), I use this sign for my unary reduction connective | characterized by the reduction rule R| in the following general form:

$$\mid \begin{bmatrix} A \\ B \end{bmatrix} \Longrightarrow A.$$

This rule may look very simple and unexpected. But look at the following application to a thought:

$$\mid \begin{bmatrix} \neg A \\ A \end{bmatrix} = \neg A.$$

A judgment of such a thought gives you a 1-dimensional *negative* result. But the reason for that is not the reduction rule R| but the entry "$\neg A$" in the first dimension of the argument!

Do we need another connective with respect to the second dimension, say "↳" with $\text{↳}\begin{bmatrix} A \\ B \end{bmatrix} \Longrightarrow B$ (R↳)? Do we need a negative version of "|", say "$\nmid$" with $\nmid \begin{bmatrix} A \\ B \end{bmatrix} \Longrightarrow \neg A$. Frege is clearly arguing against this. How can we reconstruct his argumentation in our language of analysis?

Let "$\begin{bmatrix} p \\ \neg p \end{bmatrix}$" be the syntactic representation of the thought of p. Let the "p" in the first dimension of "$\begin{bmatrix} p \\ \neg p \end{bmatrix}$" be the syntactic representation of the truth of p. Let the "$\neg p$" in the second dimension of "$\begin{bmatrix} p \\ \neg p \end{bmatrix}$" be the syntactic representation of the falsity of p. Our representation of the so-called pragmatic negation can use "↳" as a unary reduction operator which takes a 2-dimensional thought (e.g., $\begin{bmatrix} p \\ \neg p \end{bmatrix}$) as argument and picks out the second dimension (in our example "$\neg p$"): $\text{↳}\begin{bmatrix} p \\ \neg p \end{bmatrix} = \neg p$.

Frege's trouble with Kant's position regarding negative judgments is that Kant does not notice the difference between "$\updownarrow$" and "↳": $\updownarrow \begin{bmatrix} p \\ \neg p \end{bmatrix} = \begin{bmatrix} \neg p \\ p \end{bmatrix}$ vs. $\text{↳}\begin{bmatrix} p \\ \neg p \end{bmatrix} = \neg p$:

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> If we call such a transition, from a thought to its opposite [↕], negating the thought, then negating in this sense [↕] is not co-ordinate with [has not the same dimensionality like] judging [|], and may not be regarded as the polar opposite [∤?] of judging [|]; for what matters in judging [|] is always the truth [picking the first dimension as true], whereas we may pass from a thought to its opposite [via ↕] without asking which is true. To exclude misunderstanding, let it be further observed that this transition occurs in the consciousness of a thinker, whereas the thoughts that are the termini a quo and ad quem of the transition were already in being before it occurred; so that this psychical event makes no difference to the make-up and the mutual relations of the thoughts. (Frege 1960, p. 128).[30]

If we look at the ("positive") counterpart "–" of our thought-negation "↕", it does not have any effect on its argument. I.e., its "reduction" rule R– is simply

$$-\begin{bmatrix} A \\ B \end{bmatrix} \Longrightarrow \begin{bmatrix} A \\ B \end{bmatrix}$$

and, therefore $-\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix}$. We could think that negation "↕" and position "–" have nothing in common because $-\begin{bmatrix} A \\ B \end{bmatrix} \neq \updownarrow \begin{bmatrix} A \\ B \end{bmatrix}$ and also $-\begin{bmatrix} A \\ \neg A \end{bmatrix} \neq \updownarrow \begin{bmatrix} A \\ \neg A \end{bmatrix}$. But if we look at arguments of the form $\begin{bmatrix} A \\ A \end{bmatrix}$ we get $-\begin{bmatrix} A \\ A \end{bmatrix} = \updownarrow \begin{bmatrix} A \\ A \end{bmatrix}$. Of course, $\begin{bmatrix} A \\ A \end{bmatrix}$ does not represent a thought. But it shows that to get different results in applying the reduction rules R– and R↕ we need logically different expressions in the two dimensions of their arguments.

[30] „Nennt man nun ein solches Übergehen von einem Gedanken zum entgegengesetzten Verneinen [↕], so ist dieses Verneinen [↕] gar nicht gleichen Ranges mit dem Urteilen [|] und gar nicht als entgegengesetzter Pol [∤?] zum Urteilen [|] aufzufassen; denn beim Urteilen [|] handelt es sich immer um Wahrheit, wohingegen man von einem Gedanken zum entgegengesetzten übergehen kann, ohne nach der Wahrheit zu fragen [via ↕]. Um Mißverständnis auszuschließen, sei noch bemerkt, daß dieses Übergehen in dem Bewußtsein eines Denkenden geschieht, daß aber sowohl der Gedanke, von dem übergegangen wird, als auch der Gedanke, zu dem übergegangen wird, bestanden haben, bevor dies geschieht, daß also durch diesen seelischen Vorgang an dem Bestande und an den Beziehungen der Gedanken zueinander nichts geändert wird." (FREGE 1918/19b, p. 152).

The reduction rule R– reminds us of Frege's use of "⊢" as composed of "|" and "–":

$$|-\begin{bmatrix} A \\ B \end{bmatrix} = |\begin{bmatrix} A \\ B \end{bmatrix} = A \text{ in general and}$$

$$|-\begin{bmatrix} p \\ \neg p \end{bmatrix} = |\begin{bmatrix} p \\ \neg p \end{bmatrix} = p \text{ and } |-\begin{bmatrix} \neg p \\ p \end{bmatrix} = |\begin{bmatrix} \neg p \\ p \end{bmatrix} = \neg p \text{ for classical thoughts.}$$[31]

Here "–" *keeps* 2-dimensionality like "↕", but "|" *reduces* dimensionality.

The difference between "–" vs. "|" and "↕" vs. "↳" can also be observed with respect to tautologies and contradictions: $\begin{bmatrix} \top \\ \bot \end{bmatrix}$ vs. $\top$ and $\begin{bmatrix} \bot \\ \top \end{bmatrix}$ vs. $\bot$.

$$-\begin{bmatrix} \top \\ \bot \end{bmatrix} = \begin{bmatrix} \top \\ \bot \end{bmatrix} \text{vs. } |\begin{bmatrix} \top \\ \bot \end{bmatrix} = \top \updownarrow \begin{bmatrix} \top \\ \bot \end{bmatrix} = \begin{bmatrix} \bot \\ \top \end{bmatrix} \text{vs. } \Lsh\begin{bmatrix} \top \\ \bot \end{bmatrix} = \bot.$$

But with respect to tautologies and contradictions we can neglect the difference between both dimensions in the following sense:

$$\begin{bmatrix} \top \\ \bot \end{bmatrix} \text{is a tautology } (\models_{10}\begin{bmatrix} \top \\ \bot \end{bmatrix}) \text{ iff } |\begin{bmatrix} \top \\ \bot \end{bmatrix} \text{is a tautology iff} \models \top.$$

And in the full system L^2 we have

$$\models_{10}\begin{bmatrix} A \\ B \end{bmatrix} \text{iff} \models \text{A}.$$

$$\begin{bmatrix} \bot \\ \top \end{bmatrix} \text{is a contradiction } (\models_{10} -\begin{bmatrix} \top \\ \bot \end{bmatrix}) \text{ iff } ?\begin{bmatrix} \top \\ \bot \end{bmatrix} \text{is a contradiction iff } ?\bot.$$

Do we really need "|","↕" and "↳" as "Urbestandteile" (undefined basic elements) to express all these things? Do we need two ways of judgments: positive and negative? Frege says clearly "no"!

[31] There is another aspect of "-": It has to guarantee that the result of applying it to an arbitrary argument yields a thought. We will come to this point in the last section.

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Thus the assumption of two different ways of judging must be rejected. But what hangs on this decision? It might perhaps be regarded as valueless, if it did not effect an economy of logical primitives and their expressions in language. On the assumption of two ways of judging we need:

1. affirmative assertion $[|\begin{bmatrix} A \\ B \end{bmatrix} \Longrightarrow A, |\begin{bmatrix} p \\ \neg p \end{bmatrix} = p]$;
2. negative assertion, e.g. inseparably attached to the word 'false'
$$[\Lsh\begin{bmatrix} A \\ B \end{bmatrix} \Longrightarrow B, \Lsh\begin{bmatrix} p \\ \neg p \end{bmatrix} = \neg p];$$
3. a negative word like 'not' in sentences uttered non-assertively
$$[\updownarrow\begin{bmatrix} A \\ B \end{bmatrix} \Longrightarrow \begin{bmatrix} B \\ A \end{bmatrix}, \updownarrow\begin{bmatrix} p \\ \neg p \end{bmatrix} = \begin{bmatrix} \neg p \\ p \end{bmatrix}, \updownarrow\begin{bmatrix} \neg p \\ p \end{bmatrix} = \begin{bmatrix} p \\ \neg p \end{bmatrix}].$$

If on the other hand we assume only a single way of judging, we only need:

1. assertion $[|]$;
2. a negative word $[\updownarrow]$:

Such economy always shows that analysis has been pushed further, which leads to a clearer insight. There hangs together with this an economy as regards a principle of inference … (FREGE 1960, p. 130 f.)[32]

[32] „So ist denn die Annahme von zwei verschiedenen Weisen des Urteilens zu verwerfen. Aber was hängt denn von dieser Entscheidung ab? Vielleicht könnte man sie für wertlos halten, wenn dadurch nicht eine Ersparung an logischen Urbestandteilen und an dem, was ihnen sprachlich entspricht, bewirkt würde. Bei der Annahme von zwei verschiedenen Weisen des Urteilens haben wir nötig:

1. die behauptende Kraft im Falle des Bejahens $[\, |\begin{bmatrix} A \\ B \end{bmatrix} \Longrightarrow A, \ |\begin{bmatrix} p \\ \neg p \end{bmatrix} = p \,]$,

2. die behauptende Kraft im Falle des Verneinens, etwa in unlöslicher Verbindung mit dem Worte "falsch" $[\, \Lsh\begin{bmatrix} A \\ B \end{bmatrix} \Longrightarrow B, \Lsh\begin{bmatrix} p \\ \neg p \end{bmatrix} = \neg p \,]$,

3. ein Verneinungswort wie "nicht" in Sätzen, die ohne behauptende Kraft ausgesprochen werden $[\, \updownarrow\begin{bmatrix} A \\ B \end{bmatrix} \Longrightarrow \begin{bmatrix} B \\ A \end{bmatrix}, \ \updownarrow\begin{bmatrix} p \\ \neg p \end{bmatrix} = \begin{bmatrix} \neg p \\ p \end{bmatrix}, \ \updownarrow\begin{bmatrix} \neg p \\ p \end{bmatrix} = \begin{bmatrix} p \\ \neg p \end{bmatrix} \,]$.

Nehmen wir dagegen nur eine einzige Weise des Urteilens an, haben wir dafür nur nötig
1. die behauptende Kraft $[|]$,
2. ein Verneinungswort $[\updownarrow]$
Eine solche Ersparung zeigt immer eine weitergetriebene Zerlegung an, und diese bewirkt eine klarere Einsicht. Damit hängt eine Ersparung eines Schlußgesetzes zusammen.“ (FREGE 1918/19b, p. 154).

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"|" and "↕" are primitives characterized by different patterns of reduction rules. "?" can be defined and is, therefore, *not primitive*:

$$\text{↳}\begin{bmatrix} A \\ B \end{bmatrix} =_{df} |\updownarrow \begin{bmatrix} A \\ B \end{bmatrix}.$$

$$|\updownarrow \begin{bmatrix} p \\ \neg p \end{bmatrix} = \neg p$$

1.	$\lvert\updownarrow \begin{bmatrix} p \\ \neg p \end{bmatrix} = \neg p$	
2.	$\lvert \begin{bmatrix} \neg p \\ p \end{bmatrix} = \neg p$	R↕
3.	$\neg p = \neg p$	R|

Let us come back to some of Frege's questions (FREGE 1960, p. 129):

(a) "Are there two different ways of judging, of which one is used for the affirmative, and the other for the negative, answer to a question?" The answer is clearly "no!" There is only one way of judging which is neutral with respect of being positive or negative. The result of judging depends solely on the content of its argument. Judging (|) can only applied to thoughts. Correspondingly, "$|A$" is not well-formed. But we will see in a moment how "$| - A$" can be well-formed.[33]

(b) "Or is judging the same act in both cases?" Yes! In the context of the *Begriffsschrift* "|" is indicating that the whole expression following it is a *logical thought*, that Frege has a proof for it or it is itself already an axiom. Judging is simply judging. The final result of a proof can look "positive" like $\vdash p \supset p$ or "negative" like $\vdash \neg(p \wedge \neg p)$. Please look back to the move from "affirmative assertion" to "assertion" without "affirmative" in the quotation above (FREGE 1960, p. 130 f.).

(c) "Does negating go along with judging?" No! *Negating* is the transition

[33] Frege "considers the talk of negation in the sense of 'negative judgement' as logically misleading, because, in his view, any judgement is per se an affirmative judgement." (GREIMANN 2018, p. 414). We say – in accordance with Frege – that judgment is neutral with respect to negation and affirmation. Frege's formulation "we assume only a single way of judging" (1960, p. 130) does not mean "a single positive way". We can verify this by his move from "affirmative assertion" ("die behauptende Kraft im Falle des Bejahens") to simply "assertion" ("die behauptende Kraft") (ibid. & FREGE 1918/19b, p. 154). Frege is not arguing that "negative judgments" are misleading. We are on the wrong track if we think that "negative judgements" are expressible by a single connective. The definition shows that "Verneinen" means applying neutral judging to the negation of a thought. The result itself can be true or false.

from a thought (a 2-dimensional expression) to its opposite thought (its corresponding 2-dimensional expression). *Judging* takes a thought (a 2-dimensional expression) and yields a truth-value (a 1-dimensional expression by selecting the first dimension in our approach).

(d) "Or is negation part of the thought that underlies the act of judging?" We have to be precise here: "$\updownarrow$" cannot be part of expressions of the form $\begin{bmatrix} A \\ \neg A \end{bmatrix}$ or $\begin{bmatrix} \neg A \\ A \end{bmatrix}$. "$\updownarrow A$" is not well-formed. But "$\updownarrow$" is in the scope of "|". This should not be confused with our use of classical negation "¬" inside 2-dimensional expressions to represent the logical form of thoughts. "¬" has only arguments given by the language of L^1.[34]

Our starting point were seemingly exclusive answers to the (epistemo)logical status of equations by Kant and Frege: Expressions like "7 + 5 = 12" are *synthetic* for Kant but *analytic* for Frege. But that's a misleading shortening. Both thinkers agree that we need two different dimensions to characterize mathematical judgments (Kant) or sentences (Frege). They have to be not only *synthetic* in one dimension but also a priori in the other dimension for Kant. Frege's program was to show that equations are provable thoughts which cannot be perceived by senses in one dimension but they are nevertheless objective in the other dimension. Both thinkers agree that the creation of a 1-dimensional thirdness is not a solution to answer the question "How is metaphysics/logic as science possible". We have shown that Kant's *synthetic judgments a priori* as well as Frege's *thoughts* can be reconstructed as *2-dimensionally structured intermediate cases* within our language of analysis. From a methodological point of view Frege is a follower of Kant without being Kantian in motivating his thoughts philosophically and explicating them logically. Frege's way of characterizing thoughts as bivalent logical objects enables him to interpret negation as a logical connective which takes an arbitrary thought and yields its opposite thought. We say that logical negation keeps dimensionality. Judging is reducing a 2-dimensional thought

[34] Greimann writes: „... for Frege, to judge is to make a choice between opposite thoughts that contains both a positive and a negative judgement" (GREIMANN 2018, p. 411). This formulation is misleading in two respects: (a) "to judge is to make a choice between opposite thoughts": If we make a choice between two thoughts, then the result should be one of these two thoughts. But Frege says: judgment is "the recognition of the truth of a thought" (FREGE 1956, p. 294). I think "the recognition of the truth of a thought" is not a thought. If a thought has the form $\begin{bmatrix} \neg p \\ p \end{bmatrix}$, then the recognition of the truth of this thought ($|\begin{bmatrix} \neg p \\ p \end{bmatrix}$) is the truth of ¬ p. The result is a truth-value (1-dimensional) and not a thought again (2-dimensional). (b) "thoughts that contains both a positive and a negative judgement". If this were true, then we could apply "|" inside thoughts?! Thoughts "contain" both truth-values at once without any functional connection (bivalence). We realize that by taking ordered pairs consisting of A and ¬ A without using any connective.

to a 1-dimensional truth-value. Negative judging is simply judging the negation of a thought. Frege criticizes Kant for assuming that positive and negative judgments are two autonomous types. Kant was not able to see that the logical form of negative judgments is not primitive but a composition of judging with logical negation of thoughts in its scope. Frege is convinced that in his *Begriffsschrift* a final reduction to one dimension – the true – is indispensable. The judgment stroke indicates that what follows it belongs to the *Begriffsschrift*.[35] But a consequence within our language of analysis using L^2 is that tautologies remain 2-dimensional expressions which are $\models_{10}$-valid. Not the lack of the false is crucial but simply the inner order of the true ($\top$) and the false ($\bot$).

6. An Outlook: Applying our approach to Frege's *Begriffsschrift*

In the last part we use our language of analysis to try to connect it with Frege's earlier investigations in his *Begriffsschrift* with respect to *judgment*, *content* and *negation*. We are aware that the concepts "sense" and "thought" are not in the center of his interest.[36] Our remarks are a first trial and open for critical discussion.

(a) The (left) vertical stroke (the judgment stroke) & the horizontal stroke (the content stroke):

> A judgment is always to be expressed by means of the sign
>
> $$\vdash\!\!-\!\!-$$
>
> This stands to the left of the sign or complex of signs in which the content of the judgment is given. If we *omit* the little vertical stroke at the left end of the horizontal stroke, then the judgment is to be transformed into *a mere complex of ideas;* the author is not expressing his recognition or non-recognition of the truth of this. Thus, let
>
> $$\vdash\!\!-\!\!-\, A$$
>
> mean the judgment: 'unlike magnetic poles attract one another.' In that case

[35] Cf. Wittgenstein's criticism in his *Tractatus* T 4.442: "(Frege's ‚Urteilstrich' ‚$\vdash$' ist logisch ganz bedeutungslos; er zeigt bei Frege (und Russell) nur an, dass diese Autoren die so bezeichneten Sätze für wahr halten. ‚$\vdash$' gehört daher ebenso wenig zum Satzgefüge, wie etwa die Nummer des Satzes. Ein Satz kann unmöglich von sich selbst aussagen, dass er wahr ist.)" (WITTGENSTEIN 1984, p. 42).

[36] But we can find them: (a) "Accordingly, I divide all the symbols I use into *those that can be taken to mean various things* and *those that have a fully determinate sense*." (FREGE 1960, p. 1. "Alle Zeichen, die ich anwende, theile ich daher ein *in solche, unter denen man sich Verschiedenes vorstellen kann*, und *in solche die einen ganz bestimmten Sinn haben*." FREGE 1879, p. 1). "...even if a slight difference of sense is discernible, the agreement in sense is preponderant (FREGE 1960, p. 3. "Wenn man nun auch eine geringe Verschiedenheit des Sinnes erkennen kann, so ist doch die Uebereinstimmung überwiegend." FREGE 1879, p. 3) etc. (b) With respect to $-\!\!-\!\!-\, A$: "he may make inferences from this thought and test its correctness on the basis of these." (FREGE 1960, p. 2. "etwa um Folgerungen daraus zu ziehen und an diesen die Richtigkeit des Gedankens zu prüfen." FREGE 1879, p. 2) etc.

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$$\text{——}\, A$$

> will not express this judgment … (FREGE 1960, p. 1 f.).[37]

The structure "$\text{——}\, A$" represents "a mere complex of ideas" which Frege later reanalyzes as the *thought of A*. We can read "——…" as "the circumstance that …" ("der Umstand, dass") or "the proposition that" ("der Satz, dass"). We have already introduced our reduction operator "–" characterized by the reduction rule R–: $-\begin{bmatrix} A \\ B \end{bmatrix} \Longrightarrow \begin{bmatrix} A \\ B \end{bmatrix}$, i.e., that the application of "–" is redundant if the argument is already a 2-dimensional structure, a mere complex of ideas (a thought).

> Not every content can be turned into a judgment by prefixing $\vdash\!\text{——}$ to a symbol for the content; e.g. the idea 'house' cannot. Hence we distinguish contents that *are*, and contents that *are not, possible contents of judgment.*
>
> As a constituent of the sign *the horizontal stroke combines the symbols following it into a whole; assertion, which is expressed by the vertical stroke at the left end of the horizontal one, relates to the whole thus formed.* The horizontal stroke I wish to call the *content-stroke*, and the vertical the *judgment-stroke.* The content-stroke is also to serve the purpose of relating any sign whatsoever to the whole formed by the symbols following the stroke. *The content of what follows the content-stroke must always be a possible content of judgment.* (ibid., p. 2).[38]

"$\vdash\!\text{——}$" is a complex sign which consists of the parts "$|$" and "——". The task of the horizontal stroke is to secure

[37] „Ein Urtheil werde immer mit Hilfe des Zeichens

$$\vdash\!\text{——}$$

ausgedrückt, welches links von dem Zeichen oder der Zeichenverbindung steht, die den Inhalt des Urtheils angiebt. Wenn man den kleinen senkrechten Strich am linken Ende des wagerechten fortlässt, so soll dies das Urtheil in eine blosse Vorstellungsverbindung verwandeln, Vvon welcher der Schreibende nicht ausdrückt, ob er ihr Wahrheit zuerkenne oder nicht. Bedeute z. B.

$$\vdash\!\text{——}\, A$$

das Urtheil: ‚die ungleichnamigen Magnetpole ziehen sich an'; dann wird

$$\text{——}\, A$$

nicht dies Urtheil ausdrücken … (FREGE 1879, p. 1 f.)

[38] „Nicht jeder Inhalt kann durch das vor sein Zeichen gesetzte $\vdash\!\text{——}$ ein Urtheil werden, z. B. nicht die Vorstellung ‚Haus'. Wir unterscheiden daher beurtheilbare und unbeurtheilbare Inhalte.Der wagerechte Strich, aus dem das Zeichen $\vdash\!\text{——}$ gebildet ist, verbindet die darauf folgenden Zeichen zu einen Ganzen, und auf dies Ganze bezieht sich die Bejahung, welche durch den senkrechten Strich am linken Ende des wagerechten ausgedrückt wird. Es möge der wagerechte Strich Inhaltsstrich, der senkrechte Urtheilsstrich heissen. Der Inhaltsstrich diene auch sonst dazu, irgendwelche Zeichen zu dem Ganzen der darauf folgenden Zeichen in Beziehung zu setzen. Was auf den lnhaltsstrich folgt, muss immer einen beurtheilbaren Inhalt haben." (FREGE 1879, p. 2).

that what follows is a "possible content of judgment", it "must always be a possible context of judgment". From our point of view this means that the application of "–" to an expression which is not already 2-dimensional should be converted into a 2-dimensional one. E.g., Frege's "——" as well as our "–" are context-dependent. If we have "*A*", we do not know whether it is true or not. This situation can be explicated by rule R–*:

$$-A \Longrightarrow \begin{bmatrix} A \\ B \end{bmatrix}.$$

If "*B*" is different from "¬*A*", we get a content which cannot become a judgment. If "*B*" is "¬*A*", we get $\begin{bmatrix} A \\ \neg A \end{bmatrix}$. It represents a content which can become a judgment. (cf. FREGE 1960, p. 2). In the following we presuppose $B = \neg A$.[39] The curious point is that "–" plays different roles with respect to the structure of its arguments, but nevertheless we get

$$--A = -\begin{bmatrix} A \\ \neg A \end{bmatrix} = \begin{bmatrix} A \\ \neg A \end{bmatrix}.$$

(b) Negation is represented by adding a small vertical stroke to the content stroke like this:

Greimann describes this situation in the following way:

> Negation, so construed, is not a mental act, but a mathematical function that is a part of the semantic content of the sentences in which the negation stroke occurs. It is hence clear that that the negation stroke expresses the semantic type of negation, which refers to the contents of judgements. Frege's notation makes this syntactically visible by placing the negation stroke at the underside of the content stroke. (2018, p. 413).

"[P]lacing the negation stroke at the underside of the content stroke" is not the whole story:

> I call *this* small vertical stroke the *negation stroke.* The part of the horizontal stroke occurring to the right of the negation-stroke is the content-stroke of

[39] Our proposal is an alternative to the interpretation in STELZNER (1995). He says: „Ist der Inhaltsstrich der *Begriffsschrift* termbildender Operator, der aus einem Gebilde *H* mit beurteilbarem Inhalt den Terminus ‚der Umstand, daß H', ‚der Satz, daß *H*' bildet, so ist der ihm syntaktisch entsprechende Waagerechte der *Grundgesetze* zum Ausdruck einer Wahrheitsfunktion eingeführt, die genau dann den Wert wahr annimmt, wenn ihr Argument der Wert wahr ist. Ist das Argument der Wert falsch oder ein Gegenstand, der kein Wahrheitswert ist, so ist der Wert der Funktion der Wert falsch, d.h. der Waagerechte wird als Prädikator zum Ausdruck des Prädikats ‚ist wahr' eingeführt." (STELZNER 1995, p. 58 f.). Stelzner speaks about the syntactic correspondence („syntaktisch entsprechende") between the content stroke of the *Begriffsschrift* and the horizontal in Frege's *Basic Laws of Arithmetic*. He associates the content stroke with a term-forming operator and the horizontal with a truth-function which assigns true to true, false to false but also false to any third (non-propositional) object. My proposal is to look at the horizontal stroke as a kind of filter which has to secure that all vertices in Frege-trees are thoughts. It has not only to secure bivalence but also the right complexity of all vertices.

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> *A;* the part occurring to the left of the negation-stroke is the content-stroke of the negation of *A*. Here as elsewhere in our symbolism, no judgment is performed if the judgment-stroke is absent.
>
> —┬— A
>
> merely requires the formation of the idea that A is not the case, without expressing whether this idea is true. (FREGE 1960, p. 7).[40]

By the small vertical stroke the content stroke is divided into two parts which differ in scope. But they can differ also in the role they play with respect to A. Our representation of —┬— A is $- \updownarrow -A$ which yields

1.	$- \updownarrow -A$	
2.	$- \updownarrow \begin{bmatrix} A \\ \neg A \end{bmatrix}$	R—*
3.	$- \begin{bmatrix} \neg A \\ A \end{bmatrix}$	R↕
4.	$\begin{bmatrix} \neg A \\ A \end{bmatrix}$	R—

The result is still a thought. If we look at A, the final corresponding application of "|" yields $\neg A$: $| \begin{bmatrix} \neg A \\ A \end{bmatrix} = \neg A$. According to our interpretation this means that A is false. What about judging a double negation (+-- A)?

[40] „Ich nenne diesen kleinen senkrechten Strich den Verneinungsstrich. Der rechts vom Verneinungsstriche befindliche Theil des wagerechten Striches ist der Inhaltsstrich von A, der links vom Verneinungsstriche befindliche Theil dagegen ist der Inhaltsstrich der Verneinung von A. Ohne den Urtheilsstrich wird hier so wenig wie anderswo in der Begriffsschrift ein Urtheil gefällt. Afordert nur dazu auf, die Vorstellung zu bilden, dass A nicht stattfinde, ohne auszudrücken, ob diese Vorstellung wahr sei."

1.	$\vert - \updownarrow - \updownarrow - A$	
2.	$\vert - \updownarrow - \updownarrow \begin{bmatrix} A \\ \neg A \end{bmatrix}$	$\text{R}-^{*}$
3.	$\vert - \updownarrow - \begin{bmatrix} \neg A \\ A \end{bmatrix}$	$\text{R}\updownarrow$
4.	$\vert - \updownarrow \begin{bmatrix} \neg A \\ A \end{bmatrix}$	$\text{R}-$
5.	$\vert - \begin{bmatrix} A \\ \neg A \end{bmatrix}$	$\text{R}\updownarrow$
6.	$\vert \begin{bmatrix} A \\ \neg A \end{bmatrix}$	$\text{R}-$
7.	A	$\text{R}\vert$

Finally, let us come back to a point which we mentioned en passant. Frege characterizes thoughts as objects which are not perceivable by senses as well as objective things. So we *grasp* (*fassen*) thoughts. But is Frege's own access to his well-designed concept *thought* a kind of grasping? Or did he *find* the concept *thought*? Or did he *invent* this concept? Frege has a clear answer:

> This ideography, likewise, is a device invented for certain scientific purposes, and one must not condemn it because it is not suited to others. If it answers to these purposes in some degree, one should not mind the fact that there are no new truths in my work. I would console myself on this point with the realization that a development of method, too, furthers science. Bacon, after all, thought it better to invent a means by which everything could easily be discovered than to discover particular truths, and all great steps of scientific progress in recent times have had their origin in an improvement of method. (FREGE 1970, p. 6) [41]

And

> The mere invention of this ideography has, it seems to me, advanced logic. I hope that logici-

[41] „So ist diese Begriffsschrift ein für bestimmte wissenschaftliche Zwecke ersonnenes Hilfsmittel, das man nicht deshalb verurtheilen darf, weil es für andere nichts taugt. Wenn sie diesen Zwecken einigermassen entspricht, so möge man immerhin neue Wahrheiten in meiner Schrift vermissen. Ich würde mich darüber mit dem Bewusstsein trösten, dass auch eine Weiterbildung der Methode die Wissenschaft fördert. Hält es doch Baco für vorzüglicher ein Mittel zu erfinden, durch welches Alles leicht gefunden werden kann, als Einzelnes zu entdecken, und haben doch alle grossen wissenschaftlichen Fortschritte der neueren Zeit ihren Ursprung in einer Verbesserung der Methode gehabt." (FREGE 1879, p. V).

ans, if they do not allow themselves to be frightened off by an initial impression of strangeness, will not withhold their assent from the innovations that, by a necessity inherent in the subject matter itself, I was driven to make. (ibid., p. 7).[42]

From my point of view the *invention* of an interesting language of analysis using logical tools – my 2-dimensional framework – is an offer to readers who "do not allow themselves to be frightened off by an initial impression of strangeness". Further investigations have to demonstrate the fruitfulness and the range of this method to interpret philosophical as well as logical texts.

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[42] „Schon das Erfinden dieser Begriffsschrift hat die Logik, wie mir scheint, gefördert. Ich hoffe, dass die Logiker, wenn sie sich durch den ersten Eindruck des Fremdartigen nicht zurückschrecken lassen, den Neuerungen, zu denen ich durch eine der Sache selbst innewohnende Nothwendigkeit getrieben wurde, ihre Zustimmung nicht verweigern werden." (FREGE 1879, p. VII).

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