Nonlinear dynamical systems seen through the scope of the Quasi-polynomial theory

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A unified theory of nonlinear dynamical systems is presented. The unification relies on the Quasi-polynomial approach of these systems. The main result of this approach is that most nonlinear dynamical systems can be exactly transformed to a unique format, the Lotka-Volterra system. An abstract Lie algebraic structure underlying most nonlinear dynamical systems is found. This structure, based on two sets of operators obeying specific commutation rules and on a Hamiltonian expressed in terms of these operators, bears a strong similarity with the fundamental algebra of quantum physics.

From these properties, two forms of the exact general solution can be constructed for all Lotka-Volterra systems. One of them corresponds to a Taylor series in power of time. In contrast with other Taylor series solutions methods for nonlinear dynamical systems, our approach provides the exact analytic form of the general coefficient of that series. The second form of the solution is given in terms of a path integral. These solutions can be transformed back to solutions of the general nonlinear dynamical systems.

Keywords: Nonlinear systems.

I. INTRODUCTION

This series of two lectures was dedicated to the Quasi-polynomial (QP) approach to the theory of Nonlinear Dynamical Systems, i.e. systems of nonlinear ordinary differential equations that frequently appear in mathematical models in Physics, Chemistry, Biology and other scientific disciplines.

Though a great diversity of dynamical systems are used in scientific modelling, most of them can be cast in the form of systems of ordinary differential equations with polynomial or, more generally, quasi-polynomial nonlinearities. The latter are linear superposition of monomials with exponents that can be non-integers, real numbers. These monomials represent the interactions between the entities that are represented by the dynamical variables.

The Quasi-Polynomial approach relies on a special notation for writing the quasi-polynomials that appear in the vector field of a given dynamical system. Using that notation it is possible to show that any quasi-polynomial dynamical system can be exactly transformed to a Lotka-Volterra (LV) system of differential equations. This property reduces the task of solving, completely or partially, nonlinear dynamical systems to the task of solving the corresponding LV systems. Solving a LV system amounts to solve the infinite set of QP systems that are equivalent to that LV system.

We, thus, focus on the LV systems. Being a particular form of quadratic systems, they represent in fact the lowest degree of nonlinearity. We find an abstract Lie algebraic structure underlying these systems. A Hamiltonian is shown to generate the time evolution of any given LV system. We give three explicit realizations of the LV algebra. One of them is implemented with boson creation-destruction operators. In this framework, the Hamiltonian is expressed in terms of creation-destruction operators but is not Hermitian. Another realization leads to the Liouville partial derivative equation that represents the conservation of the probability density norm. This equation describes the time evolution of the probability density when the initial condition of the LV system is only known in probability. The abstract Lie algebraic of LV systems is then compared with that of linear systems. It can be seen that the difference between the two types of systems lies essentially in the different Hamiltonians that generates them.

Finally, we show that two forms of the exact general solution can be constructed for LV systems. They are based on two different realizations of the abstract LV algebra. One form of the solution is given in terms of a Taylor series in power of time. In contrast with other Taylor series solutions methods used for nonlinear dynamical systems, here we find the exact analytic form of the general coefficient of that series. The second form of the solution is given in terms of a path integral quite similarly with quantum field theory.

Finally, there remains an important open question: What happens to these solutions when the dynamical

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system is in a chaotic regime?

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II. LECTURE 1

A. Linear versus nonlinear dynamical systems

Dynamical systems are everywhere. They appear in:

Classical mechanics: Newton or Hamilton equations Fluid dynamics: Fourier modes of Navier-Stokes eqs Plasma physics and nonlinear optics: Coupled waves eqs Electronics: Kirchhoff equations of circuits Chemistry: Rate equations for chemical reactions Biology-Ecology: Populations dynamics equations Economy: rate equations for economic variables

Their general form is:

$$\frac{dX_1}{dt} = f_1(X_1(t), ..., X_n(t))$$
$$\frac{dX_2}{dt} = f_2(X_1(t), ..., X_n(t))$$
$$\vdots$$
$$\frac{dX_n}{dt} = f_n(X_1(t), ..., X_n(t)),$$

where the variable t is time and where the functions $f_i(X_1, ..., X_n)$ represent the interactions between the variables . Let us compare the main features of linear and nonlinear dynamical systems (DS):

1. Linear DS:

$$\dot{x}_i = \sum_{j=1}^{n} L_{ij} x_j \qquad i = 1, ..., n, \qquad (1)$$

with $\dot{x}_i = dx_i/dt$ and where L is a constant real nxn matrix.

The system of equations (1) is form-invariant under linear transformation:

$$x_i' = \sum_{j=1}^{n} T_{ij} x_j \qquad i = 1, ..., n. \qquad (2)$$

Indeed, transformation (2) acting on equation (1) leads to:

$$\dot{x}_i' = \sum_{j=1}^{n} L_{ij}' x_j' \qquad i = 1, ..., n, \qquad (3)$$

with:

$$L' = T \circ L \circ T^{-1}, \qquad (4)$$

where $\circ$ denotes the matrix product. Form-invariance means that equation (1) and its transformed version (3) have the same linear shape. The only difference is that the two matrices L and L' are different but they are related though relation (4).

The interest of this property is that one can find a particular matrix T that put L' into diagonal or Jordan form. From this one can easily get the general solution to equation (1). This is the origin of the efficiency of linear algebra in solving linear dynamical systems like (1).

2. Nonlinear DS:

$$\dot{x}_i = f_i(x_1(t), ..., x_n(t)) \qquad i = 1, ..., n. \qquad (5)$$

In contrast to the above linear DS, no general analytic method for solving equations (5) was until now available. This is due to infinite diversity of functional dependence that can have the nonlinear functions . As

a consequence, no form-invariance property similar to the above for linear DS was known before the advent of the quasi-polynomial theory presented in the sequel of this article. The unavailability of a general solution for most nonlinear DS led to the development of qualitative methods and approximation techniques such as linear stability analysis, bifurcation theory, Lyapunov exponents and the theory of chaotic attractors, numerical integration.

One should also keep in mind that most linear DS in scientific models are approximations of nonlinear DS.

B.   Examples of nonlinear DS:

The asymmetric Euler top and also the three waves model in plasma physics, hydrodynamics and nonlinear optics:

$$
\begin{aligned}
\dot{x}_1 &= -a_1 x_1 + \mu x_2 x_3 \\
\dot{x}_2 &= -a_2 x_2 + \beta x_1 x_3 \\
\dot{x}_3 &= -a_3 x_3 + \gamma x_1 x_2 \,.
\end{aligned}
\tag{6}
$$

The Lorentz model was derived from the Fourier modes transformation of the hydrodynamic equations of thermal convection:

$$
\begin{aligned}
\dot{x}_1 &= \sigma(x_2 - x_1) \\
\dot{x}_2 &= \rho x_2 - x_2 - x_1 x_3 \\
\dot{x}_3 &= x_1 x_2 - \beta x_3 \,.
\end{aligned}
\tag{7}
$$

The above models have nonlinearities that are at most quadratic but many other models, for example some chemical reaction rate equations, have nonlinearities with higher degree monomials or even non-integer degree monomials, i.e., quasi-monomials.

C.   Quasi-Polynomial (QP) notation of dynamical systems

Any dynamical system with polynomial nonlinearities of integer or non-integer degrees can be written in the following manner [1]:

$$
\dot{x}_i = \alpha_i x_i \sum_{j=1}^{N} A_{ij} \prod_{k=1}^{n} x_k^{B_{jk}} \qquad i = 1, ..., n \,, \tag{8}
$$

where:

- the coefficients A_{ij} are real constant entries of a $n \times N$ rectangular matrix A

- the exponents B_{ij} are real constant entries of a $N \times n$ rectangular matrix B

- the i are real constant components of a n dimensional vector

- n is the dimension of the DS (8)

- N is the total number of different (quasi-) monomials of form $\prod_{k=1}^{n} x_k^{B_{jk}}$.

It should be stressed that any polynomial or quasi-polynomial function can be written in the form of the right-hand-side of equation (8). This is due to the fact that the pre-factor x_i in the right-hand-side of (8) can be absorbed by a factor x_i^{-1} that can appear in some monomials $\prod_{k=1}^{n} x_k^{B_{jk}}$ depending on the particular form of matrix B.

In fact, equation (8) is a way to introduce a novel notation, the so-called quasi-polynomial notation, that permits to define the above matrix A of coefficients and the matrix B of exponents appearing in equation (8). As we show below, these matrices play a fundamental role in the quasi-polynomial theory of nonlinear DS.

D.   Examples of dynamical systems written in the QP notation

- The Euler asymmetric top (or three-waves model) (6) given above can be written in the QP notation (8) with:

$$
A = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \qquad \alpha = \begin{pmatrix} -a_1 \\ -a_2 \\ -a_3 \end{pmatrix} .
\tag{9}
$$

This is an example where, due to the particular form of matrix, the pre-factor x_i disappears due to cancellation with the inverses of these monomials. Here A and B are square matrices.

- The Lorenz system (7) takes the QP form (8) with:

$$
A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \rho & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \qquad \alpha = \begin{pmatrix} -\sigma \\ -1 \\ -\beta \end{pmatrix} .
\tag{10}
$$

Obviously, here A and B are rectangular matrices.

E.   Quasi-monomial transformations

The quasi-monomial transformations are defined by:

$$
\dot{X}_i = \prod_{k=1}^{n} X_k^{'C_{jk}} \qquad i = 1, ..., n \,, \tag{11}
$$

where the real constant exponents C_{ik} constitute a square $n x n$ matrix C. This matrix must be invertible. To each matrix C of GLn(R) corresponds one such transformation. The set of all these transformations forms a group under the composition product that is isomorphic to GLn(R). More precisely, to avoid singularities, we restrict our scope to equations (8) for which the variables $x_i(t)$ remain in the positive orthant.

Clearly, transformation (11) is a linear relation, expressed with matrix C, between the logarithms of the new and old variables x_i and x_i'. This particular role of the logarithm of the variables is related to the pre-factor x_i appearing in the QP format (8). Dividing the two sides of equation (8) by the pre-factor permits, indeed, to re-write (8) with the logarithmic time derivative of x_i. This fact plays a fundamental role in the QP theory.

Transformation (11) brings equation (8) into:

$$\dot{x}_i' = \alpha_i' x_i' \sum_{j=1}^{N} A_{ij}' \prod_{k=1}^{n} x_k'^{B_{jk}'} \qquad (12)$$

with :

$$\begin{aligned}
\alpha' &= C^{-1} \circ \alpha \\
A' &= C^{-1} \circ A \\
B' &= B \circ C \,,
\end{aligned} \qquad (13)$$

where $\circ$ is the matrix product.

A comparison between equations (8) and (12) shows the form-invariance of the former under transformations (11). This bears some analogy with the form-invariance of the linear DS under linear transformations we mentioned above. However, in contrast with the latter, this does not lead to any general diagonalization or Jordan form for system (12). Transformations (13) are, consequently, less powerful than the linear transformations for linear DS.

However, the transformations (11) and (13), lead to a new and important result. By choosing a particular matrix C for the transformation (11), the QP system (12) is shown to reduce to a Lotka-Volterra system as we show in the next chapter. This, in turn, brings a wealth of new results on the original system (8).

F.   Transformation of QP systems into Lotka-Volterra (LV) canonical form

Let us start with the demonstration for the particular case of QP systems (8) with $n = N$, i.e., systems where the number N of independent monomials is equal to the dimension n of the system. Such systems are, of course, by no means general but the proof of their transformation to the LV form is quite easy. The general case with is similar but needs a preliminary embedding or projection as we shall discuss after proving the case .

So, let us first consider the system with $N = n$:

$$\dot{x}_i = \alpha_i x_i \sum_{j=1}^{n} A_{ij} \prod_{k=1}^{n} x_k^{B_{jk}} \qquad i = 1, ..., n \,. \qquad (14)$$

In this case the matrix B of exponents is a square matrix (A is also square). Let us, furthermore, assume that B is invertible. We, then, perform on (14) a particular quasi-monomial transformation (11) with $C = B^{-1}$. Inserting

this form of C in equations (13) we see that:

$$\begin{aligned}
\alpha' &= B \circ \alpha = \lambda \\
A' &= B \circ A = M \\
B' &= B \circ B = I \,.
\end{aligned} \qquad (15)$$

The transformed system (12), thus, becomes in this case:

$$\dot{x}_i' = \lambda x_i' + x_i' \sum_{j=1}^{N} M_{ij} x_j' \qquad i = 1, ..., n \,. \qquad (16)$$

Due to the last equation of (15), $B' = I$, the monomials in the transformed form of (14) are reduced to just the variables x_i' themselves. Thus, the transformed version of (14) becomes the equation (16) that contains only particularly simple quadratic nonlinearities.

Expression (16) is precisely the form of the so-called Lotka-Volterra dynamical systems [2, 3]. The constant coefficient M_{ij} describes the interaction between the variables x_i' and x_j'. The M_{ij} constitute the entries of a n x n square matrix M that is given by $M = B \circ A$, i.e., the matrix product of the exponents matrix and the coefficients matrix of the original QP system (14). Similarly, the new coefficients of the linear term in (16) are the components of the vector $\lambda = B \circ \alpha$ where α is the vector of coefficients of the linear term in the original system (14).

A remarkable feature is that the quite general nonlinearity of equation (14) is reduced to the simplest form of quadratic nonlinearity in (16). A wealth of mathematical results on the LV equations exists in the literature (see for example [2, 3]). These results can be transferred to all the QP systems that are equivalent to a given LV system by performing the inverse of the transformation (11). Among them are many stability properties that can be deduced from the existence of a Lyapunov function for the LV systems.

Furthermore, all the systems having matrices A, B and vector α such that their product $B \circ A$ and $B \circ \alpha$ are identical, transform into the same LV system. We can, thus, say that they are all equivalent. They constitute an equivalence class. Since the transformation (11) with $C = B^{-1}$ is a diffeomorphism, we know that the trajectories of all these systems in phase-space are transformed by a diffeomorphism into the trajectories of the corresponding LV system. Hence, we can state the proposition that the set of all QP dynamical systems is divided into equivalence classes that are, each, characterized by one canonical member, the LV system. Let us stress that since most nonlinear dynamical systems can be brought to the QP form [4], this proposition is quite powerful.

For the more general systems (8) with $n \neq N$ we refer for more details to the article [5] where the demonstration of the transformation to the LV format has been given for all the three cases. However, as a summary we can say the following. For systems with $N > n$, one must first add $N - n$ new variables to the system (8). These variables obey:

$$\frac{dX_r(t)}{dt} = 0 \qquad r = n+1, .., N \,, \qquad (17)$$

with initial conditions $x_r(t) = 1$. Hence, these new variables are constant and equal to 1. Arbitrary powers of these (N-n) variables, $X_r(t)$, $r = n + 1, .., N$ still equal to 1, can be multiplied with the monomials $\prod_{k=1}^{n} X_k^{B_{jk}}$ that appear in (8). The previous monomials can, thus, be written $\prod_{k=1}^{N} X_k^{B_{jk}}$. In other word, the original Nxn rectangular matrix B has been completed to a square NxN matrix $\hat{B}$. If the rectangular matrix B is of maximal rank, then the completed square matrix $\hat{B}$ is always invertible. In the sequel we assume that the rank of B is always maximal. Indeed, for the cases where the rank of B is not maximal, we refer to the article [5] where it is shown that the system can be decoupled into a lower dimensional QP system with a new matrix $\tilde{B}$ with maximal rank and another lower dimensional sub-system that is linear.

The whole system (8) has, thus, been transformed into a N-dimensional QP system with $n = N$:

$$\dot{x}_i = \hat{\alpha}_i x_i \sum_{j=1}^{N} \hat{A}_{ij} \prod_{k=1}^{n} x_k^{\hat{B}_{jk}} \qquad i = 1, ..., N, \qquad (18)$$

in which, to take into account equations (17), $\hat{\alpha}$ is the old vector α completed by $N - n$ lines of zero and $\hat{A}$ is the old rectangular matrix A completed, also, by $N - n$ lines of zeroes.

Now, since system (18) has a number of monomials that is equal to the number of dimensions of the system, we can proceed as in the previous case $n = N$. The matrix B^{-1} exists and we use it in order to transform the system (18) with the help of a quasi-monomial transformation of the form:

$$\dot{x}_i = \prod_{k=1}^{N} X_k^{'\hat{B}_{jk}^{-1}} \qquad i = 1, ..., N, \qquad (19)$$

leading to:

$$\dot{x}'_i = \hat{\lambda} x'_i + x'_i \sum_{j=1}^{N} \hat{M}_{ij} x'_j \qquad i = 1, ..., N, \qquad (20)$$

where

$$\hat{M} = \hat{B} \circ \hat{A} \quad \text{and} \quad \hat{\lambda} = \hat{B} \circ \hat{\alpha}. \qquad (21)$$

However, by construction of the two new square matrices $\hat{A}$ and $\hat{B}$, and of the vector $\hat{\alpha}$, one has: $\hat{M} = \hat{B} \circ \hat{A} = B \circ A = M$ and $\hat{\lambda} = \hat{B} \circ \hat{\alpha} = B \circ \alpha = \lambda$. We thus get:

$$\dot{x}'_i = \lambda x'_i + x'_i \sum_{j=1}^{N} M_{ij} x'_j \qquad i = 1, ..., N. \qquad (22)$$

This is the same Lotka-Volterra system as in (16) but, now, in N dimensions. We have embedded the dynamics of our initially n-dimensional QP system into a N-dimensional LV system.

For the case $n > N$, we refer to the article [5] for the demonstration of the reduction to LV systems. In this case, the LV system found is again the same as in (22). But now, the embedding is replaced by a projection from the original n-dimensional phase-space into a LV system living in a smaller dimensional (N) phase-space.

In fact, in all the three cases, the dimension of the LV system is the number N of independent monomials. Moreover, obviously, the new variables x'_i in terms of which the LV system is written are these monomial themselves. This is clear by inverting relation (19). Thus a simple trick to find immediately the LV system corresponding to a given QP system is to make the list of the independent monomials appearing in it (when the pre-factor x_i has been first extracted) and to calculate the time derivative of these monomials using the original QP system. This leads, directly to the LV system (22) with M and λ respectively given by $M = B \circ A$ and $\lambda = B \circ \alpha$.

We can conclude by the following main theorem :

All QP systems $\dot{x}_i = \alpha_i x_i \sum_{j=1}^{N} A_{ij} \prod_{k=1}^{n} x_k^{B_{jk}}$, $i = 1, ..., n$, whose solutions are limited to the positive orthant, can be reduced to the LV form $\dot{x}'_i = \lambda_i x'_i + x'_i \sum_{j=1}^{N} M_{ij} x'_j$, $i = 1, ..., N$, with $M = B \circ A$ and $\lambda = B \circ \alpha$.

Furthermore, the infinite set of all the QP systems is divided into equivalence classes. Any such class contains a canonical simplest form, the LV system. Each class is characterized by the matrix M and the vector $\lambda = B \circ \alpha$ in the LV system of the class.

A last remark is that the type of nonlinearities occurring in a QP system is unessential since the LV system to which it is equivalent has always a quadratic nonlinearity.

G. Extension of the set of dynamical systems that are reducible to QP systems

The generality of the above theorem is further increased by the fact that nonlinear dynamical systems

$$\dot{x}_i = f_i(x_1(t), ..., x_n(t)) \qquad i = 1, .., n, \qquad (23)$$

that contain some nonlinear functions $f_i(x_1(t), ..., x_n(t))$ which are more general than polynomial or quasi-polynomial, can be re-casted in the form of QP systems [6, 7]. This is generally possible if the functions $f_i(x_1(t), ..., x_n(t))$ are enough regular and are themselves solutions of polynomial or quasi-polynomial differential equations. Then these last equations can be added to the original system (23) to form a higher dimensional system that will be a QP system. By the main theorem shown above this new QP system can, in turn, be transformed into a LV system. The LV quadratic nonlinearity is the ultimate reduction of nonlinearity that can be reached by exact transformations. This fact considerably enlarges the field of the quasi-polynomial approach.

H. Applications: Invariants, dimensional reduction, stability

1. Exact integrability and reduction properties of QP and LV systems

In some cases a QP system can be completely integrated or reduced to a smaller dimension. This happens when enough invariants can be found or when some transformations can decouple the system into several sub-systems, some of them being linear. This happens when, for instance, the matrix A or B are not of maximal rank. If matrix A has a rank, say r, that is smaller than n (the maximal rank) then there exists a particular quasi-monomial transformation (11) that leads to find $n - r$ invariants. If, on the contrary it is B that is of non-maximal rank, then, a quasi-monomial transformation exists that decouples the transformed system into one independent nonlinear sub-system of smaller dimension and one linear sub-system that depends on the solutions of the former. Details of these properties and others that cannot be reported here are to extensive to be reported here. They can be found in the following publications [1, 8–10]. Moreover, most of these algorithms are implemented in the computer algebra software [QPSI] written in the language MAPLE [11].

2. Stability properties of QP and LV systems

For a LV system it is always possible to find some Lyapunov function that allows for studying the stability properties of its trajectories [2, 3]. Since we have shown that most nonlinear dynamical systems can be transformed into LV form, we can transfer all the stability properties of the latter to these more general dynamical systems. This has led to a large corpus of results, too large to be exposed here, that can be found in the following articles [12–17].

III. LECTURE 2

In this lecture we now focus on the Lotka-Volterra systems in order to find some of their Lie algebraic properties and, finally, to find two forms of their general solutions. These results can, in turn, be transferred to the original quasi-polynomial systems by performing the appropriate inverse quasi-monomial transformations.

First, let us transform the LV system:

$$\dot{x}_i = \lambda_i x_i + x_i \sum_{j=1}^{N} M_{ij} x_j \qquad i = 1, ..., N, \quad (24)$$

into the $N + 1$ dimensional system

$$\dot{x}_i = x_i \sum_{j=1}^{N+1} \Omega_{ij} x_j \qquad i = 1, ..., N + 1, \quad (25)$$

where the matrix Ω is the following extension of matrix m:

$$A = \begin{pmatrix} M_{11} \dots M_{1N} & \lambda_1 \\ \dots\dots\dots\dots \\ M_{N1} \dots M_{NN} & \lambda_N \\ \dots\dots\dots\dots \end{pmatrix}, \quad (26)$$

with $X_{N+1}(t = 0) = 1$ so that $X_{N+1}(t) = 1$ for all $t > 0$.

A. Carleman infinite embedding representation of LV dynamics

The Carleman embedding [18] consists in embedding the original systems of ODEs into an infinite dimensional space in the following manner.

Define for the $N + 1$ variables of (25) the following monomials:

$$\psi(r_1, r_2, ..., r_{N+1}; t) = X_1^{r_1}(t) X_2^{r_2}(t) ... X_{N+1}^{r_{N+1}}(t), \quad (27)$$

with $r_i = 0, 1, 2, ...$ for all $i = 1, ..., N + 1$.

Then, take the time derivative of (26) and use equation (25). This gives [19]:

$$\partial_t \psi(r_1, r_2, ..., r_{N+1}; t) = H \psi(r_1, r_2, ..., r_{N+1}; t), \quad (28)$$

with the pseudo-Hamiltonian:

$$H \equiv \sum_{i=1}^{N+1} \sum_{j=1}^{N+1} r_i \Omega_{ij} D_j, \quad (29)$$

with the displacement operators D_j given by:

$$D_j f(r_1, r_2, ..., r_{N+1}; t) = f(r_1, r_2, ..., r_{N+1}; t). \quad (30)$$

Equation (28) looks like a Schrödinger equation, however, the pseudo-Hamiltonian operator is not Hermitian and, consequently, equation (28) cannot be associated to a true quantum system.

To be completely equivalent to the original LV problem (25), the equation (28) must be completed with the following particular initial conditions:

$$\psi(r_1, r_2, ..., r_{N+1}; 0) = X_1^{r_1}(0) X_2^{r_2}(0) ... X_{N+1}^{r_{N+1}}(0), \quad (31)$$

where the $X_i(0)$ are the initial values of the LV equation (25).

Let us look more algebraically at the above result. Using definition (30), let us compute the commutators $[D_i, r_j]$:

$$[D_i, r_j] = D_i \delta_{ij}. \quad (32)$$

Now let us go to the Heisenberg picture of the above pseudo-quantum problem (28-29) and calculate the time derivative of the operators $D_i(t)$ with the help of the pseudo-Hamiltonian H (29):

$$\dot{D}_i(t) = [D_i(t), H]. \quad (33)$$

This gives:

$$\dot{D}_i(t) = D_i(t) \sum_{j=1}^{N+1} \Omega_{ij} D_j(t) \qquad i = 1, ..., N+1. \quad (34)$$

Equation (34) is just the Lotka-Volterra equation (25), but written here for the operators .

This leads us to the conclusion that the LV equation (25) (or (34)) and its Carleman embedding (28) are, respectively, the Heisenberg and the Schrödinger versions of the same pseudo-quantum system. The time evolution of this system is characterized by the pseudo-Hamiltonian H given in (29).

B. Abstract Lie algebraic structure of LV systems

Now, let us generalize the above notions. We define two sets of abstract operators:

$$(A_i; i = 1, ..., N+1) \quad \text{and} \quad (B_i; i = 1, ..., N+1),$$

with the following commutation rules:

$$[A_i, A_j] = 0, \quad [B_i, B_j] = 0, \quad [B_i, A_j] = B_i \delta_{ij}. \quad (35)$$

With these operators let us define a pseudo-Hamiltonian operator:

$$H \equiv \sum_{i=1}^{N+1} \sum_{j=1}^{N+1} A_i \Omega_{ij} B_j, \quad (36)$$

where the constant coefficients Ω_{ij} are the coupling constants of the Lotka-Volterra system (25).

Then, we have the time evolution in the Heisenberg picture:

$$\dot{B}_i(t) = [B_i(t), H] = B_i \sum_{j=1}^{N+1} \Omega_{ij} B_j \qquad i = 1, ..., N+1, \quad (37)$$

and the time evolution of the same system in the Schrödinger picture:

$$\partial_t \psi(t) = H \psi(t). \quad (38)$$

The time evolution of the operators A_i can be easily calculated and obeys a system of non-autonomous linear differential equations of no interest here.

The equation (37) is just the abstract form of the LV system for the operators B_i, while the equation (38) is the abstract form of the pseudo-Schrödinger equation (28). These equations are independent of the explicit realization of the operators A_i and B_i. In fact, the operators D_i and r_i ($i = 1, ..., N+1$) were one instance of such a realization with $A_i = r_i$, $B_i = D_i$. We shall now present two others realizations of the abstract operators A_i and B_i.

C. Two other realizations of the LV abstract Lie algebra: creation-destruction operators and Liouville equation

1. Creation destruction operator realization

Let us now realize the above operators Ai and Bi in terms of boson creation-destruction operators a_i, a_i^+. The latter obey the usual boson commutation rules:

$$[a_i, a_j^+] = \delta_{ij}, \quad [a_i, a_j] = 0, \quad [a_i^+, a_j^+] = 0 \qquad i = 1, ..., N+1. \quad (39)$$

We now identify the operators A_i with the number operators $a_i^+ a_i$ and the operators B_i with the destruction operators :

$$A_i = a_i^+ a_i, \quad B_i = a_i. \quad (40)$$

With the help of the commutators (39), it is easy to check that the commutation rules (35) are verified for the A_i's and the B_i's.

The abstract pseudo-Hamiltonian (36) now becomes:

$$H \equiv \sum_{i=1}^{N+1} \sum_{j=1}^{N+1} a_i^+ a_i \Omega_{ij} a_j. \quad (41)$$

Let us calculate the time derivative of the ai(t) operators in the Heisenberg picture:

$$\dot{a}_i(t) = [a_i(t), H]. \quad (42)$$

This is readily done and gives :

$$\dot{a}_i(t) = a_i(t) \sum_{j=1}^{N+1} \Omega_{ij} a_j(t). \quad (43)$$

This is exactly the LV system for the time evolution of the destruction operators a_i, $i = 1, ..., N+1$. The time evolution of the operators $A_i = a_i^+ a_i$ is governed by linear differential equations that are not relevant here.

One could also calculate the time evolution in the Schrödinger picture but is not interesting for our next purposes.

Though very similar to quantum physical systems, the system characterized by the above pseudo-Hamiltonian (41) is not physical: It is not hermitian and does not preserve the number of "particles "in time. But, of course, we do not expect a general nonlinear dynamical system to be equivalent to a quantum mechanical system!

2. Liouvillian realization

Let us now consider the following realization of the abstract operators A_i and B_i ($i = 1, ..., N+1$):

$$A_i \equiv -\frac{\partial}{\partial X_i} X_i, \quad B_i \equiv X_i. \quad (44)$$

The commutation rules (35) can easily be verified using the elementary rules of derivative calculation.

The pseudo-Hamiltonian (36) becomes here:

$$H \equiv -\sum_{i=1}^{N+1}\sum_{j=1}^{N+1}\frac{\partial}{\partial X_i}X_i\Omega_{ij}X_j \,. \tag{45}$$

In the Heisenberg picture, the time evolution of the

operators $x_i(t)$:

$$\dot{x}_i(t) = [x_i(t), H] \qquad i = 1,...,N+1\,, \tag{46}$$

is easily shown to give :

$$\dot{x}_i(t) = x_i\sum_{j=1}^{N+1}\Omega_{ij}x_j \qquad i = 1,...,N+1\,, \tag{47}$$

which is exactly the Lotka-Volterra system.

In the Schrödinger picture, using equation (38), we get :

$$\partial_t\psi(r_1, r_2, ..., r_{N+1}; t) = -\sum_{i=1}^{N+1}\sum_{j=1}^{N+1}\frac{\partial}{\partial X_i}X_i\Omega_{ij}X_j\psi(r_1, r_2, ..., r_{N+1}; t)\,. \tag{48}$$

This partial derivative equation is easily recognizable. It is the Liouville equation associated to the LV system. It governs the time evolution of the probability density $\psi(r_1, r_2, ..., r_{N+1}; t)$ of the point representing the position at time t of the LV system in the $N+1$ phase-space. The probability density at time zero must be a given distribution function ψ_0 :

$$\psi(r_1, r_2, ..., r_{N+1}; 0) = \psi_0(r_1, r_2, ..., r_{N+1})\,. \tag{49}$$

Below, we shall use this realization in order to derive a path integral form of the general solution of a LV system.

D. The abstract Lie algebraic structure of linear and LV dynamical systems

Lets us assume we have two dynamical systems, one linear and one LV, in N+1 dimensions phase-space. We can describe both types of systems with sets of operators A_i, B_i.

For linear systems, these operators must satisfy the following commutation rules :

$$[A_i, A_j] = 0, \quad [B_i, B_j] = 0, \quad [B_i, A_j] = \delta_{ij}\,. \tag{50}$$

For LV systems, these operators must satisfy the commution rules already given in (35):

$$[A_i, A_j] = 0, \quad [B_i, B_j] = 0, \quad [B_i, A_j] = B_i\delta_{ij}\,. \tag{51}$$

For both type of systems the pseudo-Hamiltonian has the same expression in terms of these operators:

$$H \equiv \sum_{i=1}^{N+1}\sum_{j=1}^{N+1}A_i\Omega_{ij}B_j\,. \tag{52}$$

It can easily be checked, using respectively commutation rules (50) and (51), that the Heisenberg time evolution

equations for the $N+1$ operators Bi:

$$\dot{B}_i(t) = [B_i(t), H]\,, \tag{53}$$

yields:

Linear system :

$$\dot{B}_i(t) = \sum_{j=1}^{N+1}\Omega_{ij}B_j\,. \tag{54}$$

LV system :

$$\dot{B}_i(t) = B_i\sum_{j=1}^{N+1}\Omega_{ij}B_j\,. \tag{55}$$

Thus, the only difference in the Lie algebra of both types of systems resides in the third commutator in both expressions (50) and (51). The evaluate the generality of this result, one should keep in mind that the LV system represents, in fact, a whole class of equivalence of nonlinear dynamical systems as we have shown earlier in this presentation.

E. General solution via Taylor series

In order to derive the Taylor series solution of the LV system (25), the most appropriate realization is that generated by the operators r_i and D_i in equations (27-34). Indeed, let us first combine the equations (28) and (29). This gives:

$$\partial_t\psi(r_1, r_2, ..., r_{N+1}; t) = \sum_{i=1}^{N+1}\sum_{j=1}^{N+1}r_i\Omega_{ij}D_j\psi(r_1, r_2, ..., r_{N+1}; t)\,, \tag{56}$$

with the particular initial condition:

$$\psi(r_1, r_2, ..., r_{N+1}; 0) = X_1^{r_1}(0) X_2^{r_2}(0) ... X_{N+1}^{r_{N+1}}(0) \, . \quad (57)$$

The Taylor series solution of (56) is expressed by:

$$\psi(\bar{r}, t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} c_k(\bar{r}) \, , \quad (58)$$

where $\bar{r}$ denotes the vector $(r_1, ..., r_{N+1})$ and the coefficients of the series, $c_k(\bar{r})$, are given by:

$$c_k(\bar{r}) = \frac{d^k}{dt^k} \psi(\bar{r}, t)|_{t=0} \, , \quad (59)$$

which, with the help of equation (56) iterated k time, leads to:

$$c_k(\bar{r}) = \left(\sum_{i=1}^{N+1} \sum_{j=1}^{N+1} r_i \Omega_{ij} D_j \right)^k \psi(\bar{r}; 0) \, , \quad (60)$$

that is, the k-th power of the Hamiltonian acting on the initial ψ function.

Using the definition (30) of the displacement operators D_i and the expression (57) of the initial function, one gets without difficulty:

$$c_k(\bar{r}) = [\prod_{j=1}^{N+1} X_j^{r_j}(0)] \sum_{i_1=1}^{N+1} \cdots \sum_{i_k=1}^{N+1} (r\Omega)_{i_1} ((r\Omega)_{i_2} + (\Omega)_{i_1 i_2}) ... (((r\Omega)_{i_k} + (\Omega)_{i_1 i_k} + ... + (\Omega)_{i_{k-1} i_k}) X_{i_1}(0) ... X_{i_k}(0) \, . (61)$$

To obtain the Taylor series solution of the LV system itself, one must remember that

$$X_i = \psi(\bar{r} = e^i, t) \, , \quad (62)$$

where

$$e^i = (0, ..., 0, r_i = 1, 0, ...0) \, . \quad (63)$$

Thus, the Taylor series solution of the LV system:

$$x_i = \sum_{j=1}^{N+1} \Omega_{ij} x_j \, , \quad (64)$$

is given by :

$$X_i(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} c_k(e^i) \, , \quad (65)$$

with the following expression of the general coefficient :

$$c_k(e^i) = X_j(0) \sum_{i_1=1}^{N+1} \cdots \sum_{i_k=1}^{N+1} (r\Omega)_{i_1} ((r\Omega)_{j_2} + (\Omega)_{i_1 i_2}) ... (((r\Omega)_{i_k} + (\Omega)_{i_1 i_k} + ... + (\Omega)_{i_{k-1} i_k}) X_{i_1}(0) ... X_{i_k}(0) \, .$$

$$(66)$$

The solution (65, 66) of the LV system (64) defines a new class of special functions [19]. This is in complete analogy with the generalized hypergeometric functions [20] that are defined as Taylor series giving the solutions of second order linear differential equations.

The fact, that we have obtained an analytic expression of the general term of the Taylor series gives much more information on the solution than the usual numerical Taylor expansions that are currently used to solve nonlinear differential systems. First, it allows for developing a computer algebra software for solving non-numerically

LV systems [21]. The program computes and stores results that are dependent of the symbolic values of the coefficients. This permits to vary the parameters at will without re-beginning the whole calculations.

Also, the knowledge of the general term of a Taylor series permits to evaluate the radius of convergence, to calculate analytical continuations and, also, to derive asymptotic expansions for the solution. However, this last claim is only in its infancy and must be developped.

One should, however, recognize that applications of this form of the solution are up to now limited by the

difficulty of finding closed expressions of the many finite sums appearing in the expression (66) of the Taylor coefficients. The complexity of these sums reflects all the complexity of the nonlinear dynamical systems! It is comparable to the complexity of the partition function of a system of N particles interacting through a binary interaction potential, at thermal equilibrium. The partition function is also expressed in terms of many finite sums of a summand that depends on the binary interaction potential between the N particles and its exact closed expression is usually out of our present technical ability.

However, expression (66) provides an alternative to other methods that could help obtaining more informations on the behaviour of the solutions of nonlinear dynamical systems.

F. General solution via path integrals

The Liouvillian realization of the abstract LV algebra is better adapted for the objective of deriving a path integral form of the solution of the LV system (25).

We recall the description of the time evolution of the system in Schrödinger picture:

$$\partial_t \psi(X_1, X_2, ..., X_{N+1}; t) = -\sum_{i=1}^{N+1}\sum_{j=1}^{N+1}\frac{\partial}{\partial X_i}X_i\Omega_{ij}X_j\psi(X_1, X_2, ..., X_{N+1}; t). \tag{67}$$

For the Liouville equation (67) to be completely equivalent to the original deterministic LV equation (25), we need to assume that the initial distribution is deterministic, i.e., the initial condition is known with probability one :

$$\psi(X_1, X_2, ..., X_{N+1}; 0) = \prod_{i=1}^{N+1}\delta(X_i - X_i^0). \tag{68}$$

The formal solution of equation (67), for any initial distribution, can always be written as :

$$\psi(X_1, X_2, ..., X_{N+1}; t) = <\prod_{i=1}^{N+1}\delta(X_i - X_i(t, X_1(0), ..., X_{N+1}(0))) >_{\psi(t=0)}, \tag{69}$$

where the symbol $<>_{\psi(t=0)}$ means the operation of average over the initial distribution $\psi(\overline{X}; t = 0)$.

The explicit expression of the average in (69) gives:

$$\psi(X_1, X_2, ..., X_{N+1}; t) = \int dX_1(0)...dX_{N+1}\prod_{i=1}^{N+1}\delta(X_i - X_i(t, X_1(0), ..., X_{N+1}(0)))\psi(X_1(0), X_2(0), ..., X_{N+1}(0); t = 0), \tag{70}$$

where $X_i(t, X_1(0), ..., X_{N+1}(0))$, $(i = 1, ..., N + 1)$, denotes the trajectory of the solution to (25).

Let us now consider the expression (70) for a short time $\Delta t \equiv t/n$:

$$\psi(X_1, X_2, ..., X_{N+1}; \Delta t) = \int dX_1(0)...dX_{N+1}\prod_{i=1}^{N+1}\delta(X_i - X_i(\Delta t, X_1(0), ..., X_{N+1}(0)))\psi(X_1(0), X_2(0), ..., X_{N+1}(0); t = 0). \tag{71}$$

For large enough values of n, that is, for small enough

values of , the solution of the LV system (25) can be

written:

Inserting equation (72) into (71), we get:

$$\psi(X_1, X_2, ..., X_{N+1}; \Delta t) = \int dX_1(0)...dX_{N+1}(0) \times$$

$$\prod_{i=1}^{N+1} \delta(X_i - X_i(0) - \Delta t X_i(0) \times$$

$$\sum_{j=1}^{N+1} \Omega_{ij} X_i(0)) \psi(X_1(0), X_2(0), ..., X_{N+1}(0); t = 0).$$

$$\tag{73}$$

$$X_i(\Delta t, X_1(0), ..., X_{N+1}(0)) = X_i(0) + \Delta t X_i(0) \sum_{j=1}^{N+1} \Omega_{ij} X_i(0) \tag{72}$$

$i = 1, ..., N+1$.

If we displace the initial time from $t = 0$ to $t = \Delta t$ in (73), we obtain:

$$\psi(X_1, X_2, ..., X_{N+1}; 2\Delta t) = \int dX_1(1)...dX_{N+1}(1) \prod_{i=1}^{N+1} \delta(X_i - X_i(1) - \Delta t X_i(1) \sum_{j=1}^{N+1} \Omega_{ij} X_i(1)) \psi(X_1(1), X_2(1), ..., X_{N+1}(1); \Delta t), \tag{74}$$

where we used the notation:

The relation (74) can be translated up to time $t = n\Delta t$:

$$\psi(X_1, X_2, ..., X_{N+1}; n\Delta t) = \int dX_1(n-1)...dX_{N+1}(n-1) \times$$

$$\prod_{i=1}^{N+1} \delta(X_i - X_i(n-1) - \Delta t X_i(n-1) \sum_{j=1}^{N+1} \Omega_{ij} X_i(n-1)) \times$$

$$\psi(X_1(n-1), X_2(n-1), ..., X_{N+1}(n-1); (n-1)\Delta t). \tag{76}$$

The recursion on n appearing in (75) can now be iterated down to $n = 0$, leading to:

$$X_i(k) \equiv X_i(k\Delta t). \tag{75}$$

$$\psi(X_1, X_2, ..., X_{N+1}; n\Delta t) = \int d^{N+1} X_1(n-1)...d^{N+1} X_{N+1}(0) \times$$

$$\prod_{i_{n-1}=1}^{N+1} \delta(X_{i_{n-1}} - X_{i_{n-1}}(n-1) - \Delta t X_{i_{n-1}}(n-1) \sum_{j_{n-1}=1}^{N+1} \Omega_{i_{n-1} j_{n-1}} X_{j_{n-1}}(n-1)) \times$$

$$\prod_{i_{n-2}=1}^{N+1} \delta(X_{i_{n-2}}(n-2) - X_{i_{n-2}}(n-2) - \Delta t X_{i_{n-2}}(n-2) \sum_{j_{n-2}=1}^{N+1} \Omega_{i_{n-2} j_{n-2}} X_{j_{n-2}}(n-2))...$$

$$\prod_{i_0=1}^{N+1} \delta(X_{i_0}(1) - X_{i_0}(0) - \Delta t X_{i_0}(0) \sum_{j_0=1}^{N+1} \Omega_{i_0 j_0} X_{j_0}(0)) \psi(X_1(0), X_2(0), ..., X_{N+1}(0); 0). \tag{77}$$

Obviously, the n integrals appearing in (76) represent a summation over all the possible (broken) paths connecting the point $\overline{X}(0)$ to the point $\overline{X}(n\Delta = t) \equiv \overline{X}$ in the N+1-dimensional phase-space of the LV system (25). In the limit $n \to \infty$, $\Delta t = 0$, $n\Delta = t$ (t finite), these paths become smooth curves and the n integrals tend towards a so-called path integral.

As a last step, we introduce the particular initial con-

dition (68) in the expression (76). This enables us to perform the integral over all the possible positions of the initial point $\overline{X}(0)$. This operation along with the following property of the Dirac delta distribution:

$$\delta(ax) = \frac{1}{|a|}\delta(x) \tag{78}$$

transforms the expression (76) into:

$$\psi(X_1, X_2, ..., X_{N+1}; n\Delta t) = \frac{1}{\Delta t^{n(N+1)}} \int d^{N+1}X_1(n-1)...d^{N+1}X_{N+1}(0) \times$$

$$\prod_{i_{n-1}=1}^{N+1} \delta(\frac{X_{i_{n-1}} - X_{i_{n-1}}(n-1)}{\Delta t} - X_{i_{n-1}}(n-1) \sum_{j_{n-1}=1}^{N+1} \Omega_{i_{n-1}j_{n-1}} X_{j_{n-1}}(n-1)) \times$$

$$\prod_{i_{n-2}=1}^{N+1} \delta(\frac{X_{i_{n-2}}(n-2) - X_{i_{n-2}}(n-2)}{\Delta t} - X_{i_{n-2}}(n-2) \sum_{j_{n-2}=1}^{N+1} \Omega_{i_{n-2}j_{n-2}} X_{j_{n-2}}(n-2))...$$

$$\prod_{i_0=1}^{N+1} \delta(\frac{X_{i_0}(1) - X_{i_0}^0}{\Delta t} - X_{i_0}^0 \sum_{j_0=1}^{N+1} \Omega_{i_0 j_0} X_{j_0}^0). \tag{79}$$

One observes in (78) the apparition of the component i of the discretized velocity at time $k\Delta t$: $(X_i(k) - X_i(k-1))/\Delta t$. Taking the limit $n \to \infty$, $\Delta t = 0$, $n\Delta = t$ (t finite) of (78), we obtain the path integral form of the solution to the LV system (25):

$$\psi(\overline{x}, t) = \int_{\overline{x}(0)}^{\overline{x}=\overline{x}(t)} Dx(\cdot) \prod_{i=1}^{N+1} \delta(\dot{x}_i(\cdot) - x_i(\cdot) \sum_{j=1}^{N+1} \Omega_{ij} x(\cdot)), \tag{80}$$

where the path integral operator is defined as:

$$\lim_{n \to \infty, \Delta t=0, n\Delta=t} \frac{1}{\Delta t^{n(N+1)}} \int d^{N+1}X_1(n-1)...d^{N+1}X_{N+1}(0), \tag{81}$$

and where $x_i(\cdot)$ represents the function x_i of time as a point in a set of functions.

The path integral (79) is a new result. It is a very complex mathematical object. Usually, in order to compute such path integrals, one needs to go back to the discrete time version. Here, it means to return to equation (78). After completing the calculations, one takes the limit $n \to \infty$, $\Delta t = 0$, $n\Delta = t$. There is no guaranty up to now that such calculations are feasible but I think them worth to be attempted.

Of course all the results exposed in the two last chapters can be back transformed to the original QP systems. We do not present this inverse transformation here because it is quite simple.

IV. CONCLUSION

I hope to have convinced the reader that the QP theory is a significant progress in the study of nonlinear dynamical systems. It unifies the all domain around the study of the unique Lotka-Volterra canonical form. This offers the possibility to transfer all the mathematical knowledge about LV systems to the much larger domain of the nonlinear dynamical systems that can be put in the quasi-polynomial format. This transfer is far from being completed by now but it would be of high interest to achieve it because the amount of results known on the LV systems is really huge.

Furthermore, I have shown that the LV format leads to a deeper understanding of the Lie algebraic structure that lies underneath the linear and LV systems, and more generally nonlinear dynamical systems. This new, more algebraic, way of looking at dynamical systems seems promising. It provides a new flexibility by allowing for choosing among various realizations of the same dynamical system, the one that is more adapted to one's objective.

The algebraic point of view permits also to discover a deep similarity between dynamical systems that are, after all, classical systems and quantum systems. Among the quantum tools that can be applied to classical dynamical systems we have the different pictures of the time evolution of a dynamical system such as the Heisenberg and Schrödinger pictures. We have found also that a dynamical system is always generated by a Hamiltonian operator. When written in terms of creation-destruction boson operators, the similarity of these Hamiltonians with those of quantum systems becomes striking.

Among the various realizations of the algebra of LV systems we showed that two of them, at least, are enough practical to obtain a form of the general solution. These solutions are quite complex. This is expected for nonlinear dynamical systems that can, for instance, be in a chaotic regime.

A deep question remains: What happens to these solutions when a given dynamical system is in such a chaotic regime? The answer to this question needs a way to extract from one or another form of the general solution its time asymptotic behaviour. This is a very exciting work that still remains to be done!

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