



Lagrangian Chaotic Mixing in a Nonlinear Model of Resistive Drift-wave Turbulence in Tokamak Plasmas

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In fusion plasma, numerical simulations are commonly employed to investigate the confinement properties of plasma in the bulk region of tokamaks. The modified Hasegawa-Wakatani (MHW) equations are used to model the behavior of the plasma, which enables us to understand the radial transport in two-dimensional numerical simulations of electrostatic resistive drift-wave turbulence. By utilizing the MHW equations, we have gained insights into the low-to-high confinement (L-H) transitions that occur spontaneously in the plasma when it moves from a low confinement stage, characterized by turbulent flow, to a turbulence-suppressed regime known as zonal flow. To investigate these transitions, we vary the value of a control parameter α , which is related to adiabaticity, in numerical simulations, and observe the transition between the two regimes. This simplified model of L-H transitions can provide valuable information for tokamaks. To identify the Lagrangian coherent structures (LCS) and to better characterize the chaotic mixing during the L-H transition, we computed the finite-time Lyapunov exponent (FTLE) of the calculated velocity field derived from the electrostatic potential. We further compared the statistics of the chaotic mixing of the two regimes. The results of our study offer insights into the turbulent transport processes in magnetic confinement fusion plasmas.

Keywords: modified Hasegawa-Wakatani (MHW), Low-to-high confinement (L-H), Lagrangian coherent structures (LCS), finite-time Lyapunov Exponent (FTLE), probability distribution functions (PDFs).

I. INTRODUCTION

The world's energy dependency has been rapidly increasing due to global population growth and industrialization. One raising concern, as a result, is the energy supply, with fossil fuel being a finite resource and a major contributor to climate change. There is an escalating

need for a clean, safe, carbon-neutral, and politically neutral form of electricity generation [1]. Nuclear fusion has been recognized as an alternative solution for the energy dependency problems, as well as for climate change.

Nuclear fusion in nature occurs in the interior of stars. The Sun is powered by fusion reactions, which is when the nuclei fuse together, and produces a mass that is less than the mass of the reactants combined. This small mass loss is due to the energy that is released [1]. There are two main ways to recover the energy from the fusion collision: magnetic confinement and inertial fusion [2]. A tokamak is a fusion reactor that focuses on extracting

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this energy using magnetic confinement.

In a more detailed explanation of how magnetic confinement works, it has been explained that the process takes advantage of the charge of plasma particles and attempts to design a magnetic field to confine plasma [1]. A plasma is an ionized gas or a quasineutral gas of charged and neutral particles which exhibits collective behavior [2]. As such, the plasma serves as the fuel that is required for fusion to occur.

When researching fusion plasma, the turbulent processes are a significant challenge, particularly in the radial transport at the edge of a tokamak [3]. A fundamental aspect of fusion research is to comprehend the dynamics of the turbulent radial flux of particles and heat in magnetized plasmas, as it can lead to enhancements in the confinement properties of fusion devices, including tokamaks [4]. Numerical simulations are a useful tool to model this behavior and determine the overall plasma confinement properties in the bulk region. The Hasegawa-Wakatani equations facilitate understanding of the radial transport through two-dimensional numerical simulations of electrostatic resistive drift-wave turbulence.

We carried out numerical simulations to gain a better understanding of low-to-high confinement (L-H) transitions, a phenomenon observed in fusion plasmas. The L-H transitions occur spontaneously as the plasma transits from a low confinement stage, known as turbulent flow, to a turbulence-suppressed regime referred to as zonal flow, which is associated with the high confinement stage. This transition holds immense significance due to its ability to improve the confinement, and therefore studying L-H transitions presents an opportunity for confinement enhancement [5]. In addition, we compute the finite-time Lyapunov exponent (FTLE) to detect the Lagrangian coherent structures (LCS) and, as a result, provide a more comprehensive characterization of the chaotic mixing during the L-H transition.

This paper is organized as follows. Section II describes the model employed and the numerical tools. The numerical results are presented in Section III, and the conclusion is given in Section IV.

II. COMPUTATIONAL METHODS

A. Modified Hasegawa-Wakatani (MHW) equations

We executed numerical simulations using a simplified model of a tokamak plasma, which accounts for the influence of the zonal component, based on the modified Hasegawa-Wakatani (MHW) equations [5]

$$\frac{\partial}{\partial t}\zeta + \{\varphi, \zeta\} = \alpha(\tilde{\varphi} - \tilde{n}) - D\nabla^4\zeta, \quad (1)$$

$$\frac{\partial}{\partial t}n + \{\varphi, \zeta\} = \alpha(\tilde{\varphi} - \tilde{n}) - \kappa\frac{\partial\varphi}{\partial y} - \nabla^4n, \quad (2)$$

where the physical setting of the model is of a tokamak plasma in a constant magnetic field equilibrium $B = B_0\nabla z$, and a nonuniform density $n_0 = n_0(x)$ in the edge region. The equations (1) and (2) contain parameters $\{a, b\}$ which denote the Poisson bracket, n representing the density fluctuations, and the ion vorticity $\zeta \equiv \nabla^2\varphi$ which is a 2D Laplacian depending on the electrostatic potential (φ). The background density $\kappa \equiv (\partial/\partial x)\ln n_0$ has an unchanging exponential profile and is constant, while D represents the dissipation coefficient. The adiabaticity operator α is set as a constant coefficient in this physical configuration [5].

The MHW equations are obtained by subtracting the zonal components from the resistive coupling term, which results in $\alpha(\varphi - n)$ becoming $\alpha(\tilde{\varphi} - \tilde{n})$. The velocity field equations are derived from the electrostatic potential (φ) [5]

$$v_x \equiv -\frac{\partial\tilde{\varphi}}{\partial y}, \quad (3)$$

$$v_y \equiv \frac{\partial\tilde{\varphi}}{\partial x}. \quad (4)$$

The particle density flux Γ_r , is a correlation between the particle density (n) and radial velocity ($v_r = -\partial\tilde{\varphi}/\partial y$) [4]

$$\Gamma_r = \langle nv_r \rangle. \quad (5)$$

In this model the radial direction is represented by the x direction, therefore $v_r \equiv v_x$.

B. Finite-time Lyapunov Exponents (FTLE)

The analysis of chaotic mixing properties of fluids is carried out through the computation of the finite-time Lyapunov Exponent (FTLE) and subsequent observation of the resulting Lagrangian coherent structures (LCS) that emerge from the velocity field. The LCS can be defined as ridges within the FTLE fields, which represent special gradient lines transverse to the direction of minimum curvature [6].

The FTLE is defined as a finite time average of the maximum expansion rate for a pair of particles that are advected in the flow [6]. Another definition, describes the Lyapunov exponent as a measure of the sensitivity of a fluid particle's future behavior [7]. To define the FTLE mathematically, we must first consider the evolution of a perturbed point $\vec{x}' = \vec{x} + \delta\vec{x}$, where $\delta\vec{x}$ is infinitesimal and arbitrary oriented. After an interval τ , this perturbation becomes, $\delta\vec{x} = \vec{x}' - \vec{x}$,

$$\delta\vec{x}_\tau = \phi_{t_0}^{t_0+\tau}(\vec{x}') - \phi_{t_0}^{t_0+\tau}(\vec{x}), \quad (6)$$

where the flow map is denoted as $\phi_{t_0}^{t_0+\tau}$ and can be expanded into a Taylor series in the neighborhood of \vec{x} ,

$$\begin{aligned} \phi_{t_0}^{t_0+\tau}(\vec{x}') \Big|_{\vec{x}} &= \phi_{t_0}^{t_0+\tau}(\vec{x}) \\ &+ \frac{d\phi_{t_0}^{t_0+\tau}}{d\vec{x}} \Big|_{\vec{x}} (\vec{x}' - \vec{x}) + \dots \\ &\approx \phi_{t_0}^{t_0+\tau}(\vec{x}) + \frac{d\phi_{t_0}^{t_0+\tau}}{d\vec{x}} \Big|_{\vec{x}} (\vec{x}' - \vec{x}). \end{aligned} \quad (7)$$

The finite-time Cauchy-Green deformation tensor is defined as,

$$C_{t_0}^{t_0+\tau} = \left(\frac{d\phi_{t_0}^{t_0+\tau}}{d\vec{x}} \right)^\tau \left[\frac{d\phi_{t_0}^{t_0+\tau}}{d\vec{x}} \right]. \quad (8)$$

The eigenvectors of $C_{t_0}^{t_0+\tau}$ as $\vec{\xi}_1$ and $\vec{\xi}_2$, with corresponding eigenvalues $\lambda_1 > \lambda_2$ satisfying,

$$C_{t_0}^{t_0+\tau} \vec{\xi}_i = \lambda_i \vec{\xi}_i, \quad i = 1, 2, \quad (9)$$

and $\|\vec{\xi}_i\| = 1$. In addition, it is supposed that the perturbation $\delta\vec{x}$ is aligned with $\vec{\xi}_1$ (being the maximum deformation),

$$\delta\vec{x}_0 = \|\delta\vec{x}_0\| \vec{\xi}_1. \quad (10)$$

From Equations (9) and (10),

$$\delta\vec{x}_\tau = \sqrt{\lambda_1} \|\delta\vec{x}_0\|. \quad (11)$$

Finally, the definition of the FTLE can be derived from Eq. 11 [6],

$$\sigma_{t_0}^{t_0+\tau}(\vec{x}) = \frac{1}{|\tau|} \ln \sqrt{\lambda_1}, \quad (12)$$

where the largest FTLE is given by Eq. 12 and can be computed for both positive and negative integration times (τ) due to the absolute value operation. The positive integration time is a forward-time integration that reveals a repelling LCS [6].

III. SIMULATION RESULTS

To solve the MHW equations, we employ a finite differences method with a grid resolution of 256x256, implemented via a Fortran code. For computing the FTLE, we use a C code with a grid resolution of 1024x1024. The obtained results from both methods are then visualized and analyzed using MATLAB, which facilitates the computation of probability density functions (PDFs) values. This approach provides an understanding of the dynamics underlying the MHW and FTLE phenomena.

A. Modified Hasegawa-Wakatani (MHW)

In Figure 1, we can see the electrostatic potential (ϕ) obtained from a simulation using a Fortran code. This simulation used the finite differences method to solve the MHW equations. The resulting solutions were imported into MATLAB with a 256x256 grid resolution for a detailed representation of the electrostatic potential.

The left panel of Figure 1 displays the patterns of ϕ in the turbulent regime, where as the right panel of Fig. 1 shows the patterns of ϕ in the zonal flow regime. The patterns were obtained by setting the value of the adiabaticity parameter $\alpha = 0.010$ (turbulent regime) and $\alpha = 0.018$ (zonal flow regime).

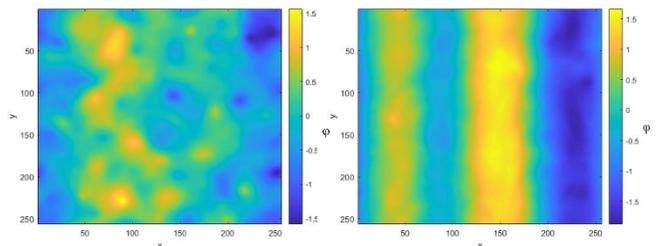


Figure 1. The electrostatic potential in the turbulent regime (left panel) and zonal flow (right panel).

A transition from L-H regimes in tokamaks can be modeled by varying the control parameter α , related to adiabaticity [5]. This simplified model, depicted in the left and right panels of Figure 1, provides insight into the behavior of tokamaks under different confinement conditions. The left panel represents low confinement, while the right panel represents high confinement.

Figure 1 illustrates that the zonal flow in tokamaks exhibits a zonally elongated structure of ϕ . This structure arises due to the Kelvin-Helmholtz instability of the drift waves, which effectively suppresses drift wave activity [5]. As a result, the zonal flow has high confinement properties, making it an important factor in understanding plasma turbulence and confinement in tokamaks.

Table I. Numerical values of Γ_r for the two regimes

Turbulent regime	1.4110
Zonal flow	-0.07577

Table 1 presents the computation of the radial flux (Eq. 5) for the turbulent and zonal flow regimes across the entire simulation domain. The results display that the turbulent regime exhibits a higher radial flux value than the zonal flow regime. This observation is expected, since the elongated patterns of the zonal flow act as transport barriers for the flow.

The patterns of ϕ depicted in Fig. 1 can give a hint of

the presence of coherent structures. For example, vortices are related to regions of localized minima and maxima of ϕ . These regions can be easily recognized in the turbulent regime. However, the detection of structures based on snapshots of velocity fields (i.e. an Eulerian approach) can give misleading results [7]. Coherent structures can be objectively detected using a Lagrangian approach.

B. Finite-time Lyapunov Exponents (FTLE)

Figure 2 depicts the detected LCS obtained from the FTLE ($\sigma_{t_0}^{t_0+\tau}$) using velocity fields. These solutions are then imported into MATLAB to create the images shown in Figure 2, with a 1024x1024 grid resolution, allowing for a detailed representation of the LCS.

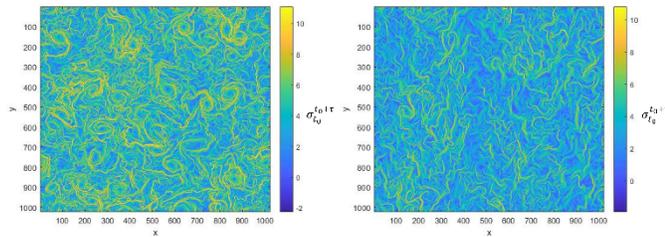


Figure 2. The FTLE in the turbulent regime (left panel) and zonal flow (right panel).

Figure 2 provides a more detailed understanding of the locations of the transport barriers (LCS). The barriers are displayed using a color gradient, with stronger barriers appearing in yellow and weaker ones in blue. When comparing both images (in Figure 2), it becomes evident that a higher number of strong barriers are present in the turbulent regime (left panel), as to be expected.

While contrasting Figures 1 and 2, it is evident that Figure 2 provides a clearer visualization of the locations of the barriers formed. The overlapping images, from Figure 1 into Figure 2, reveal that not every localized maximum or minimum of ϕ from Figure 1 corresponds to a vortex. Moreover, the definition of vortex boundaries using ϕ patterns can be a difficult task. These boundaries are clearly marked by ridges of the FTLE field. This observation suggests that the FTLE is more effective at detecting vortices when compared to visual inspection of the electrostatic potential.

The probability distribution functions (PDFs) of the FTLE, as depicted in Figure 2, are presented in Figure 3. Broad PDFs have been linked to heterogeneous mixing in previous studies [8]. The PDFs in Figure 3 reveal that the turbulent regime has a more heterogeneous mixing pattern than the zonal flow regime. This is evident from the broader PDFs of the turbulent regime than in comparison to the corresponding PDFs of the zonal flow.

The difference in mixing can be attributed to the fact that the zonal flow is a regime of high confinement that suppresses turbulent transport [5],

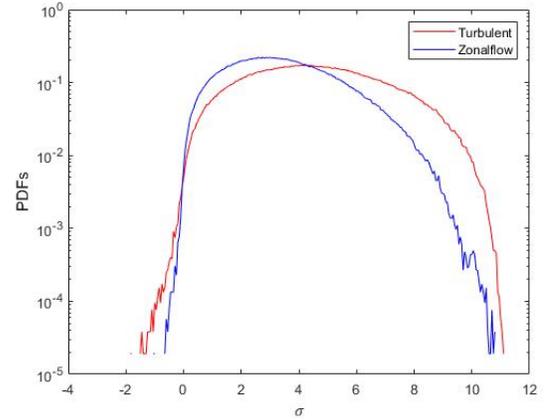


Figure 3. PDFs of (Red) Turbulent regime versus (Blue) Zonal flow.

IV. CONCLUSION

In summary, our study focused on numerical simulations of the modified Hasegawa-Wakatani equations, examining two distinct regimes: one characterized by turbulence dominance and another characterized by zonal flow. To gain insight into the flow's turbulent mixing properties, we employed finite-time Lyapunov exponents (FTLE), a commonly used tool for Lagrangian analysis of turbulent fluids. Our analysis, based on the construction of probability distribution functions (PDFs), led to the conclusion that the turbulent regime displayed a more heterogeneous mixing behavior than the zonal flow regime, which is consistent with the high-confinement regime associated with the zonal flow. The techniques and insights gained from our study may aid in the comprehension of drift-wave induced turbulence in tokamak plasmas.

The computation of the FTLE represents a simple technique for the detection of Lagrangian coherent structures based on ridges of the resulting field. It has been demonstrated that ridges of the FTLE field can lead to inconsistent results [9]. For this reason we are currently applying more advanced techniques such as geodesic theory [10] for a future work.

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