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# Effective Potential for binary mistures of a Bose-Einsten

Valéria de Carvalho Souza
*Departamento Cosmo*
*Centro Brasilei de Pesquisas Físicas (CBPF),*
*Rio de Janeiro, RJ, Brazil.*

It has a degree, a master's degree and a doctorate from the University of the State of Rio de Janeiro (UERJ). He has postdated at the Brazilian Center for Physical Research (CBPF). Has experience in the field of condensed matter physics, phase transition to finite temperature such as condensation of Bose-Eistein. He developed a study on BCS-BEC crossover (CBPF). His research is now centered on the study of gauge theories for gravitation, effective gauge theories with violation of Lorentz symmetry, and effects of spin-orbit interaction on a more fundamental physics via the weakly relativistic boundary of the Dirac equation.

Keywords: Bose-Einstein, Mean Field

## I. INTRODUCTION

Multi-component quantum gases are fascinating systems [1]. Basic research in this area has enormously grown in the last years [2]. Due to the ability of optically trapping and cooling gases to extremely low temperatures, it is possible to study different phenomena in bosonic [3, 4] as well as fermionic mixtures [5]. Important quantum effects like Bose-Einstein condensation (BEC) and superconductivity can now be studied in a very controlled way in multicomponent atomic systems.

Interesting experiments with mixed bosonic quantum fluids have been done by simultaneously trapping $^{87}Rb$ atoms in two different hyperfine states [6–9]. The relative population is reached by applying a coupling field characterized by a Rabi frequency $\Omega_R$ and a detuning $\nu$ with respect to the spacing between the energy levels of the two hyperfine states. In this way, it is possible to transfer atoms from one hyperfine state to the other, producing a Josephson-type interaction between species [10–12].

In general, the name "Josephson interaction" refers to the interaction of a large number of bosonic degrees of freedom allowed to occupy two different quantum states. Although it was originally proposed in superconductors systems [13], where the bosons are Cooper pairs, there are many other systems where this effect shows up. A review covering different physical systems can be find in Ref. [14]. We can distinguish two types of Josephson effects[15]: the so called "external", where the two states are spatially separated, like, for instance, in BEC trapped in a double well potential [16–19], or the "internal", where the two bosonic states are interpenetrated, without geometrical distinction, like, for instance, the experiments in Refs. [7, 8]. In this paper, we are mainly interested in the later case of internal Josephson-type interactions.

Static and dynamical properties of binary bosonic mixtures in different trap geometries have been studied theoretically by essentially using Gross-Pitaevsky equations [20–23]. Moreover, to study properties of uniform condensates, specially those issues related with fluctuations, such as symmetry restoration, re-entrances, etc, quantum field theory at finite density and temperature [24] is a useful technique. Related models, such as $O(N)$ models, have also been extensively studied by using large $N$ approximation and renormalization group techniques [25, 26]. These papers are mostly concentrated in multicomponent systems which conserve the particle number of each species independently.

Motivated by these results, we decided to address the effect of Josephson-like interactions in uniform bosonic mixtures. For simplicity, we have considered an $O(2)$ model, perturbed with an explicit symmetry breaking term parametrized by the Rabi frequency $\Omega_R$ and the detuning term $\nu$. This model is analyzed in mean-field approximation plus Gaussian fluctuations. We show that, in a definite temperature regime, the two atomic species condensate at the same critical temperature $T_c$ and their relative phase is locked by the phase of the detuning parameter $\nu$. On the other hand, the relative population of each condensate strongly depends on the ratio $\Omega_R/|\nu|$. We have found that, due to the original $O(2)$ symmetry, the effective Rabi frequency, given by $\Omega_{\rm eff} = \sqrt{\Omega_R^2 + |\nu|^2}$ is strongly renormalized by thermal fluctuations. However, the ratio $\Omega_R/|\nu|$ remains unaffected under quantum as well as thermal fluctuations. Interestingly, we have found a re-entrance of the $O(2)$ symmetry at a very small temperature in which $\Omega_{\rm eff}(T_r) = 0$.

The paper is organized as follows. In section II we describe a general model for a binary mixture using quantum field theory language. In § III, we concentrate on the $O(2)$ model perturbed with Josephson interactions. In section IV we present the mean field solution, while in § V we analyze the effect of fluctuations. Numerical results are presented in section VI and, finally, we discuss our results in § VII.

## II. A QUANTUM FIELD THEORY FOR BINARY BOSONIC MIXTURES

We will consider two bosonic species described by two complex fields, $\phi(\vec{x},t)$ and $\psi(\vec{x},t)$. The model is defined by the action

$$S=\int d^3xdt\ \{\mathcal{L}_\psi+\mathcal{L}_\phi+\mathcal{L}_I\}\,, \tag{1}$$

where $\mathcal{L}_\psi$ and $\mathcal{L}_\phi$ are the non-relativistic quadratic Lagrangian densities

$$\mathcal{L}_\psi=\psi^*\left(i\partial_t-\frac{\nabla^2}{2m}+\mu_\psi\right)\psi\,, \tag{2}$$

$$\mathcal{L}_\phi=\phi^*\left(i\partial_t-\frac{\nabla^2}{2m}+\mu_\phi\right)\phi\,. \tag{3}$$

$\mu_\phi=\mu+\Omega_R$ and $\mu_\psi=\mu-\Omega_R$ are the chemical potentials. Thorough the paper, we have used a unit system in which $\hbar=1$. $\mu$ controls the overall particle density at the time that the Rabi frequency $\Omega_R$ controls the population imbalance. We chose the same mass for each species, since we are interested in mixtures composed by a single element in two different hyperfine states.

The interaction Lagrangian density $\mathcal{L}_I$ can be split in two terms,

$$\mathcal{L}_I=\mathcal{L}_c+\mathcal{L}_J\,. \tag{4}$$

The first term, $\mathcal{L}_c$, contains two-body interactions that preserves the particle number of each species individually. For diluted gases, it can be approximated as a local quartic polynomial of the form

$$\mathcal{L}_c=-\frac{g_\psi}{2}\left(\psi^*\psi\right)^2-\frac{g_\phi}{2}\left(\phi^*\phi\right)^2-g_{\phi\psi}\psi^*\psi\phi^*\phi, \tag{5}$$

where the coupling constants $g_\psi=4\pi a_\psi/m$, $g_\phi=4\pi a_\phi/m$ and $g_{\phi\psi}=8\pi a_{\phi\psi}/m$ are written in terms of the intra-species s-wave scattering lengths $a_\psi$, $a_\phi$ and the inter-species s-wave scattering length $a_{\phi\psi}$. Note that this interaction term is invariant under $U(1)_\phi\otimes U(1)_\psi$ transformation.

The second term of Eq. (4) does not conserve particle number of each species individually, however, it conserves the total particle number. This term explicitly breaks the symmetry of Eq. (5) as $U(1)_\phi\otimes U(1)_\psi\to U(1)_{\phi+\psi}$. We generally call these terms as Josephson interactions, since they couple the phases of each bosonic component. The simplest terms can be written as

$$\mathcal{L}_J=\nu\psi^*\phi+\nu^*\phi^*\psi-\frac{g_J}{2}\left(\psi^*\psi^*\phi\phi+\phi^*\phi^*\psi\psi\right). \tag{6}$$

The quadratic term, proportional to $\nu$, and the quartic two-body interaction term have in general very different origins. The one-particle term is proportional to the detuning $\nu$. For generality, we have considered a complex parameter in such a way to control the relative phases of the condensates. Considering the two species as components of an iso-spin doublet, this term arises like an effective spin-orbit interaction. Of course, we could also consider one-body terms of this type with derivative couplings [27], however, to keep matters as simple as possible we will consider only this term. The second term in Eq. (6), represents scattering processes where the internal hyperfine state of the atoms is not conserved. In the absence of $\nu$, these processes are unlikely since both hyperfine states are energetically well separated. However, in the presence of a laser with small detuning between the frequency differences, a very small coupling constant $g_J$ could produce qualitatively different results.

Some aspects of the phase diagram of the model of Eqs. (2), (3) and (5), without Josephson couplings ($\mathcal{L}_\mathcal{J}=0$) was previously studied in Ref. [24]. The zero temperature mean-field analysis provides constraints over the coupling constants in order to have Bose-Einstein condensates. In particular, it is simple to show that

$$g_\phi g_\psi-g_{\psi\phi}^2>0 \tag{7}$$

in order to have two coexisting condensates. If the condition (7) is not satisfied, both condensates cannot coexist and they tend to spatially separate as observed in Ref. [28]. Interestingly, quantum as well as thermal fluctuations do not change this general behavior [24]. However, the presence of Josephson couplings qualitatively changes the phase diagram of this model.

## III. $O(2)$ MODEL WITH JOSEPHSON ANISOTROPY

The model described in the preceding section has a very reach phase diagram depending on the relative values of the coupling constants and temperature. However, there is a special point of maximum symmetry where the analysis gets simpler. Thus, let us analyze model (1-6) in its maximum symmetry point given by $g_{\phi,\psi}=g_\phi=g_\psi=g$, $\Omega_R=0$, $\nu=0$ and $g_J=0$. At this point, the interaction term, Eq. (5), takes the simpler form

$$\mathcal{L}_c=-\frac{g}{2}\left(\psi^*\psi+\phi^*\phi\right)^2. \tag{8}$$

Thus, in addition to $U(1)_\phi\otimes U(1)_\psi$ phase symmetry, there is an emergent $O(2)$ symmetry, corresponding with rotations in the iso-spin space $(\phi,\psi)$. On the one hand,

the particle number of each species is independently conserved. On the other hand, the two species are indistinguishable since any iso-spin rotation mixing the two species has exactly the same action.

Next, we minimally break the $O(2)$ symmetry by considering a term proportional to $\Omega_R$ and a Josephson term of the form,

$$\mathcal{L}_\nu = \nu\psi^*\phi + \nu^*\phi^*\psi. \tag{9}$$

For simplicity, we ignore two-body Josephson interactions (given by the term proportional to $g_J$ in Eq. (6)) since, in principle, it is of higher order than the one-body interaction term we are considering.

The structure of this model is clearly visualized by defining new fields $(\varphi_1, \varphi_2)$ obtained by a rotation of the original fields $(\phi, \psi)$,

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = M \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \tag{10}$$

where the rotation matrix is

$$M = \begin{pmatrix} \frac{\Omega_{\text{eff}}-\Omega_R}{D} & -\frac{\nu}{D} \\ \\ \frac{\nu^*}{D} & \frac{\Omega_{\text{eff}}-\Omega_R}{D} \end{pmatrix}, \tag{11}$$

with

$$D = \sqrt{\left(\Omega_{\text{eff}} - \Omega_R\right)^2 + |\nu|^2}. \tag{12}$$

and $\Omega_{\text{eff}} = \sqrt{\Omega_R^2 + |\nu|^2}$. Of course, one can immediately check that $\det(M) = 1$. With this transformation the Lagrangian density takes the form

$$\begin{aligned} \mathcal{L} = \varphi_1^* \left( i\partial_t - \frac{\nabla^2}{2m} + \mu_+ \right) \varphi_1 + \\ + \varphi_2^* \left( i\partial_t - \frac{\nabla^2}{2m} + \mu_- \right) \varphi_2 - \\ - \frac{g}{2} \left(\varphi_1^*\varphi_1 + \varphi_2^*\varphi_2\right)^2, \end{aligned} \tag{13}$$

where $\mu_\pm = \mu \pm \Omega_{\text{eff}}$. We see that, while the terms proportional to $\nu$ and $\Omega_R$ in Eq. (9) break $O(2)$ and $U(1)_\phi \otimes U(1)_\psi$ symmetries, the system still has an $U(1)_{\varphi_1} \otimes U(1)_{\varphi_2}$ symmetry in the new variables. Thus, there is a direction in the iso-spin space in which the particle number of both species is conserved independently. This simple behavior is a consequence of the $O(2)$ symmetry of the two-body interaction term, Eq. (8). It is not difficult to realize that, in the case of imbalanced two-body interactions $g_{\psi\phi} \neq g$, a term proportional to $\varphi_1\varphi_1\varphi_2^*\varphi_2^*$ would be generated upon iso-spin rotation, breaking in this way $U(1)_{\varphi_1} \otimes U(1)_{\varphi_2} \to U(1)$. In this sense, the model of Eq. (13) implements a minimal perturbation of the complex $O(2)$ model.

Interestingly, Eq. (13) does not depend on $\Omega_R$ and $\nu$ independently, but only depends on the effective Rabi frequency $\Omega_{\text{eff}} = \sqrt{\Omega_R^2 + |\nu|^2}$. On the other hand, the rotation matrix Eq. (11) depends only on the ration $\Omega_R/|\nu|$. It is instructive to see the form of the rotation matrix in two different limits.

Lets consider, for instance, $|\nu| \ll \Omega_R$. In this case,

$$M = \begin{pmatrix} \frac{|\nu|}{2\Omega_R} & -e^{i\alpha_\nu} \\ \\ e^{-i\alpha_\nu} & \frac{|\nu|}{2\Omega_R} \end{pmatrix}, \tag{14}$$

where we defined $\nu = |\nu|\exp(i\alpha_\nu)$. In the extreme limit of $\nu \to 0$ the two species are decoupled, as expected, and the mixture is proportional to $\frac{|\nu|}{2\Omega_R} + O((|\nu|/2\Omega_R)^2)$.

In the opposite limit, $|\nu| \gg \Omega_R$,

$$M = \begin{pmatrix} \frac{1}{\sqrt{2}}\left(1 - \frac{\Omega_R}{2|\nu|}\right) & -\frac{e^{i\alpha_\nu}}{\sqrt{2}}\left(1 + \frac{\Omega_R}{2|\nu|}\right) \\ \\ \frac{e^{-i\alpha_\nu}}{\sqrt{2}}\left(1 + \frac{\Omega_R}{2|\nu|}\right) & \frac{1}{\sqrt{2}}\left(1 - \frac{\Omega_R}{2|\nu|}\right) \end{pmatrix}. \tag{15}$$

In the extreme limit $\Omega_R \to 0$, the fields are simetrically superposed, depending just on the phase of the detuning parameter,

$$\varphi_1 = \frac{1}{\sqrt{2}}\left(\phi - e^{i\alpha_\nu}\psi\right), \tag{16}$$

$$\varphi_2 = \frac{1}{\sqrt{2}}\left(e^{-i\alpha_\nu}\phi + \psi\right) \quad . \tag{17}$$

Small values of $\Omega_R$ produce corrections of order $\Omega_R/|\nu|$.

## IV. MEAN-FIELD APPROXIMATION

Let us analyzed the model of Eq. (13) in mean-field approximation. This conduces to the Gross-Pitaevski equations

$$\left(i\partial_t - \frac{\nabla^2}{2m} + \mu_+ - g\left(\varphi_1^*\varphi_1 + \varphi_2^*\varphi_2\right)\right)\varphi_1 = 0, \tag{18}$$

$$\left(i\partial_t - \frac{\nabla^2}{2m} + \mu_- - g\left(\varphi_1^*\varphi_1 + \varphi_2^*\varphi_2\right)\right)\varphi_2 = 0. \tag{19}$$

Looking for uniform and static solutions $\varphi_{1,2}(x,t) \equiv \varphi_{1,2}^0$ we have

$$\left(\mu_+ - g\left[|\varphi_1^0|^2 + |\varphi_2^0|^2\right]\right)\varphi_1^0 = 0, \tag{20}$$

$$\left(\mu_- - g\left[|\varphi_1^0|^2 + |\varphi_2^0|^2\right]\right)\varphi_2^0 = 0. \tag{21}$$

Supposing that $\varphi_{1,2}^0 \neq 0$, we can subtract eq. (21) from eq. (20), obtaining $\Delta\mu = \mu_+ - \mu_- = 0$. Therefore, the two fields $\varphi_{1,2}$ cannot condensate simultaneously, since a solution $\varphi_{1,2}^0 \neq 0$ does not exist except in the case $\Delta\mu = 2\Omega_{\text{eff}} = 0$. Instead, we have two possible solutions,

$$\varphi_1^0 = 0 \qquad , \qquad |\varphi_2^0|^2 = \mu_-/g \tag{22}$$

or

$$|\varphi_1^0|^2 = \mu_+/g \qquad , \qquad \varphi_2^0 = 0 \tag{23}$$

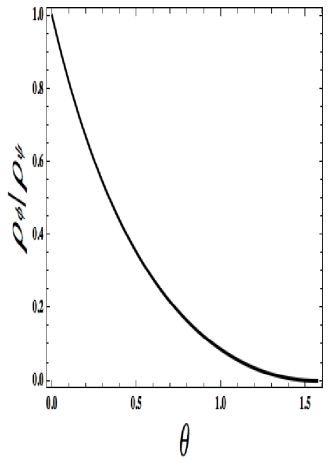


Figura 1: $|\phi_0|^2/|\psi_0|^2$ given by Eq. (29) as a function of $\theta$

Let us consider the solution $\varphi_2^0 = 0$, Eq. (23). Using the matrix $M^{-1}$, given by the inverse of Eq. (11), it is simple to turn back to the original fields, obtaining

$$\phi_0 = \frac{\Omega_{\text{eff}} - \Omega_R}{\sqrt{\left(\Omega_{\text{eff}} - \Omega_R\right)^2 + |\nu|^2}}\,\varphi_1^0, \tag{24}$$

$$\psi_0 = -\frac{\nu^*}{\sqrt{\left(\Omega_{\text{eff}} - \Omega_R\right)^2 + |\nu|^2}}\,\varphi_1^0\,, \tag{25}$$

where $\phi_0$ and $\psi_0$ are the condensate amplitudes of the fields $\phi(x)$ and $\psi(x)$, respectively. The first observation is that the two original species $\phi$ and $\psi$ condense simultaneously and the relative phase between these condensates, $\Delta\alpha$, is fixed by the phase of the parameter $\nu$,

$$\Delta\alpha = \alpha_\nu + \pi. \tag{26}$$

Moreover, the condensate fractions of the two species depends on the ratio $\Omega_R/|\nu|$. It is instructive to parametrize $\Omega_R$ and $|\nu|$ in the following way,

$$\Omega_R = \frac{\Delta\mu}{2}\sin\theta, \tag{27}$$

$$|\nu| = \frac{\Delta\mu}{2}\cos\theta, \tag{28}$$

with $0 \leq \theta \leq \pi/2$. In terms of this parametrization, the ratio between the condensate densities takes the form

$$\frac{|\phi_0|^2}{|\psi_0|^2} = \sec^2\theta\,(1-\sin\theta)^2\,, \tag{29}$$

which does not depend on $\Delta\mu$ but only on $\tan\theta = \Omega_R/|\nu|$. We depict this function in Fig. (1). For $\theta \to 0$, or $\Omega_R \to 0$ with $|\nu| \neq 0$, the two condensates have essentially the same fraction. On the other hand, for $\theta \to \pi/2$, or $|\nu| \to 0$ with $\Omega_R \neq 0$, only one of the fields condensates.

One important question is how thermal fluctuations affect the general structure of the mean-field zero temperature result. To see this, let us consider the effective potential

$$U_{eff}(\bar\varphi_1, \bar\varphi_2) = U_0(\bar\varphi_1, \bar\varphi_2) + \Delta U(\bar\varphi_1, \bar\varphi_2, T), \tag{30}$$

where $\bar\varphi_{1,2}$ represent the order parameters, $U_0$ is the bare potential and $\Delta U$ codifies the effect of fluctuations. Two important symmetries constraint the general form of this potential. In the one hand, $U(1)\otimes U(1)$ symmetry implies that the effective potential is a function of $|\bar\varphi_1|^2$ and $|\bar\varphi_2|^2$. On the other hand, since $O(2)$ symmetry is broken by the parameter $\Delta\mu = \mu_+ - \mu_-$, the potential should be a function of $\chi_s = |\bar\varphi_1|^2 + |\bar\varphi_2|^2$ and $\chi_a = |\bar\varphi_1|^2 - |\bar\varphi_2|^2$ independently. The subindexes $s$ and $a$ stands for symmetric and antisymmetric, respectively. Then,

$$U_{eff} = U_0(\chi_s, \chi_a) + \Delta U(\chi_s, \chi_a, T). \tag{31}$$

Minimizing the effective potential and assuming that $\bar\varphi_{1,2} \neq 0$, we immediately find

$$\mu_+ + \frac{\partial\Delta U}{\partial\chi_s} - g\left(|\bar\varphi_1|^2 + |\bar\varphi_2|^2\right) + \frac{\partial\Delta U}{\partial\chi_a} = 0 \tag{32}$$

$$\mu_- + \frac{\partial\Delta U}{\partial\chi_s} - g\left(|\bar\varphi_1|^2 + |\bar\varphi_2|^2\right) - \frac{\partial\Delta U}{\partial\chi_a} = 0. \tag{33}$$

These equations imply that

$$\Delta U(T) = -\frac{1}{2}\Delta\mu(T)\;\chi_a + \mathcal{F}(\chi_s, T), \tag{34}$$

where $\mathcal{F}$ is an arbitrary function. Since there is no reason to expect a linear function $\Delta U \sim \chi_a$, we conclude that, in this model, both fields $\varphi_1$ and $\varphi_2$ cannot condensate simultaneously, provided $\Delta\mu(T) \neq 0$. By carefully studying thermal fluctuations, we will see in the next section that, depending on density ratio, it could be a reentrance of the $O(2)$ symmetry, $\Delta\mu(T_r) = 0$, at extremely low temperatures $T_r \ll T_c$, changing the mean-field structure solution.

## V. EFFECT OF FLUCTUATIONS

To study thermal as well as quantum fluctuations we start by considering the following Euclidean finite temperature field theory,

$$\begin{aligned} S_{\text{E}}(\beta) = \int_0^\beta d\tau \int d^3x \Bigg[ & \varphi_1^*\left(\partial_\tau - \frac{\nabla^2}{2m} - \mu_+\right)\varphi_1 + \\ & + \varphi_2^*\left(\partial_\tau - \frac{\nabla^2}{2m} - \mu_-\right)\varphi_2 - \\ & + \frac{g}{2}\left(\varphi_1^*\varphi_1 + \varphi_2^*\varphi_2\right)^2\Bigg]\,, \end{aligned} \tag{35}$$

with $\beta = 1/T$. The partition function reads

$$\begin{aligned} Z(\beta, \vec J) &= \int \mathcal{D}\varphi_1\mathcal{D}\varphi_1^*\mathcal{D}\varphi_2\mathcal{D}\varphi_2^*\; e^{-S_{\text{E}} + \int d^3x d\tau \vec J\cdot\vec\varphi} \\ &= e^{-\beta V W[\beta, J]}, \end{aligned} \tag{36}$$

where we have introduced a source $\vec{J}$ in order to compute field correlation functions. The functional integration measure implicitly contains the cyclic bosonic boundary condition in Euclidean time $\varphi_{1,2}(0,x)=\varphi_{1,2}(\beta,x)$. $W[\beta,\vec{J}]=-\frac{1}{\beta V}\ln Z$ is the Helmholtz free energy density.

The main purpose of this section is to compute $W[\beta,J]$ in mean-field approximation plus Gaussian fluctuations. As discussed in the last section, we expect that, at least in a certain temperature region to be determined later, fluctuations will not change the general mean field structure. With this in mind, in order to compute $W[\beta,J]$ we replace in Eq. (36) the following decomposition

$$\varphi_1(x,\tau)=\varphi_1^0+\tilde{\varphi}_1(x,\tau) \tag{37}$$
$$\varphi_2(x,\tau)=\tilde{\varphi}_2(x,\tau) \tag{38}$$

in which $\int d^3x\ \tilde{\varphi}_{1,2}=0$ and $\varphi_1^0(J)$ is a solution of

$$\left.\frac{\delta S_E}{\delta\varphi_1}\right|_{\varphi_1=\varphi_1^0,\varphi_2=0}=J\ , \tag{39}$$
$$\left.\frac{\delta S_E}{\delta\varphi_2}\right|_{\varphi_1=\varphi_1^0,\varphi_2=0}=0\ . \tag{40}$$

where we have chosen $\vec{J}=cte.$ in the $\varphi_1$ direction.

Retaining up to second-order terms in the fluctuations we obtain

$$Z(\beta)=e^{-\beta V U_0(\varphi_1^0)}\int[\mathcal{D}\tilde{\varphi}]\ e^{-\int d\tau d^3x\sum_{ij}\tilde{\varphi}_i^* S_{ij}^{(2)}\tilde{\varphi}_j}\ , \tag{41}$$

where

$$U_0=-\mu_+|\varphi_1^0|^2+\frac{g}{2}|\varphi_1^0|^4\ . \tag{42}$$

The integration measure is

$$[\mathcal{D}\tilde{\varphi}]=\mathcal{D}\tilde{\varphi}_1\mathcal{D}\tilde{\varphi}_1^*\mathcal{D}\tilde{\varphi}_2\mathcal{D}\tilde{\varphi}_2^* \tag{43}$$

and the quadratic kernel

$$S_{ij}^{(2)}=\left.\frac{\delta^2 S_{\rm E}}{\delta\varphi_j^*\delta\varphi_i}\right|_{\varphi_1=\varphi_1^0,\varphi_2=0}\ , \tag{44}$$

with $i,j=1,2$.

Integrating out quadratic fluctuations we find an expression for the free energy density

$$W[J,\beta]=U_0+\Delta W \tag{45}$$

with

$$\Delta W[J,\beta]=\frac{1}{2}\ln\det\hat{S}^{(2)}\ . \tag{46}$$

The matrix $\hat{S}^{(2)}$ in the $(Re(\tilde{\varphi}_1),\ Im(\tilde{\varphi}_1),\ Re(\tilde{\varphi}_2),\ Im(\tilde{\varphi}_2))$ bases, decouples in two independent $2\times2$ blocks,

$$\hat{S}^{(2)}=\begin{pmatrix}\hat{S}_a^{(2)} & 0\\ 0 & \hat{S}_b^{(2)}\end{pmatrix}, \tag{47}$$

with

$$\hat{S}_a^{(2)}=\begin{pmatrix}-\frac{\nabla^2}{2m}-\mu_++3g|\varphi_1^0|^2 & i\partial_\tau\\ -i\partial_\tau & -\frac{\nabla^2}{2m}-\mu_++g|\varphi_1^0|^2\end{pmatrix} \tag{48}$$

and

$$\hat{S}_b^{(2)}=\begin{pmatrix}-\frac{\nabla^2}{2m}-\mu_-+g|\varphi_1^0|^2 & i\partial_\tau\\ -i\partial_\tau & -\frac{\nabla^2}{2m}-\mu_-+g|\varphi_1^0|^2\end{pmatrix}. \tag{49}$$

It is not difficult to compute the determinant in Fourier space, obtaining

$$\Delta W=\frac{1}{2\beta}\sum_{n=-\infty}^{+\infty}\int\frac{d^3q}{(2\pi)^3}\ln\left\{\left(\omega_n^2+E_1^2\right)\left(\omega_n^2+E_2^2\right)\right\}, \tag{50}$$

where $\omega_n=2\pi n\beta$ are the Matzubara frequencies,

$$E_1=\sqrt{\left(\frac{q^2}{2m}-\mu_++3g|\varphi_1^0|^2\right)\left(\frac{q^2}{2m}-\mu_++g|\varphi_1^0|^2\right)} \tag{51}$$

and

$$E_2=\frac{q^2}{2m}-\mu_-+g|\varphi_1^0|^2\ . \tag{52}$$

Summing up the Matzubara frequencies, using

$$\frac{1}{\beta}\sum_n\ln(\omega_n^2+E_i^2)=E_i+\frac{2}{\beta}\ln\left(1-e^{-\beta E_i}\right), \tag{53}$$

we obtain

$$\Delta W=\frac{1}{2}\int\frac{d^3q}{(2\pi)^3}\sum_i\left\{E_i+\frac{2}{\beta}\ln\left(1-e^{-\beta E_i}\right)\right\}. \tag{54}$$

It is interesting to note that, if we substitute the mean-field value for $\varphi_1^0$, given by Eq. (23), into Eqs. (51) and (52), we immediately obtain

$$\tilde{E}_1=\sqrt{\left(\frac{q^2}{2m}\right)\left(\frac{q^2}{2m}+2g|\varphi_1^0|^2\right)} \tag{55}$$

and

$$\tilde{E}_2=\frac{q^2}{2m}+\Omega_R^2+|\nu|^2\ . \tag{56}$$

Eqs. (55) and (56) are the usual energy excitations computed in the Bogoliuvov approximation. Note that $\lim_{q\to0}\tilde{E}_1=0$, corresponding with the Goldstone mode associated with the spontaneous breakdown of $U_{\varphi_1}(1)$ symmetry, while Eq. (56) is a gapped mode corresponding to non-condensate fluctuations.

It is useful to express the free energy $W(\beta,J)$ in terms of the order parameter

$$\bar{\varphi}=\delta W/\delta J=\varphi_1^0+\frac{1}{2}Tr\left[\frac{1}{\hat{S}^{(2)}}\frac{\delta\hat{S}^{(2)}}{\delta\varphi_1^0}\frac{\delta\varphi_1^0}{\delta J}\right]\ . \tag{57}$$

At mean-field level, the order parameter is exactly the mean-field solution $\varphi_1^0$. However, when fluctuations are taken into account, the result given by Eq. (57) is more involved.

We define the Gibbs free energy as a functional of the order parameter $\bar{\varphi}$ by making a Legendre transformation

$$\Gamma[\beta,\bar{\varphi}] = \bar{\varphi}J - W\,, \tag{58}$$

where $\delta\Gamma/\delta\bar{\varphi} = J$. In Eq. (58), $J$ is a function of the order parameter $\bar{\varphi}$ obtained by inverting Eq. (57). To leading order in the fluctuations the result is

$$\Gamma[\beta,\bar{\varphi}] = \mu_+|\bar{\varphi}|^2 - \frac{g}{2}|\bar{\varphi}|^4 - \frac{1}{2}\int\frac{d^3q}{(2\pi)^3}\sum_i\left\{E_i + \frac{2}{\beta}\ln\left(1-e^{-\beta E_i}\right)\right\} \tag{59}$$

This is the Gibbs free energy computed at mean field plus Gaussian fluctuations or, in the language of quantum field theory, the finite temperature one-loop effective action.

The actual condensate amplitude $\bar{\varphi}_m$ is computed by minimizing the free energy

$$\left.\frac{\partial\Gamma[\beta,\bar{\varphi}]}{\partial\bar{\varphi}}\right|_{\bar{\varphi}=\bar{\varphi}_m} = 0. \tag{60}$$

Using eq. (59) and defining an effective chemical potential $\bar{\mu}_+ = g|\bar{\varphi}_m|^2$, we find an expression for $\bar{\mu}_+$ in terms of the original $\mu_+$ and temperature

$$\bar{\mu}_+ = \mu_+ - \frac{1}{2}\int\frac{d^3q}{(2\pi)^3}\times \times\left\{\frac{(2\frac{q^2}{2m}+\bar{\mu}_+)(1+2n(E_1))}{\sqrt{\left(\frac{q^2}{2m}\right)\left(\frac{q^2}{2m}+2\bar{\mu}_+\right)}} + 1 + 2n(E_2)\right\}, \tag{61}$$

where $n(E_i)$ is the usual Bose distribution

$$n(E_i) = \frac{1}{e^{\beta E_i - 1}} \tag{62}$$

with $i = 1, 2$.

The total particle density of each species is given by

$$\rho_{\varphi_1} = \left.\frac{\partial\Gamma}{\partial\mu_+}\right|_{\bar{\varphi}_m} = \frac{\mu_+}{g} - \frac{1}{2}\int\frac{d^3q}{(2\pi)^3}\frac{q^2/2m}{E_1}(1+2n(E_1)) - \frac{1}{2}\int\frac{d^3q}{(2\pi)^3}(1+2n(E_2))\,, \tag{63}$$

$$\rho_{\varphi_2} = \left.\frac{\partial\Gamma}{\partial\mu_-}\right|_{\bar{\varphi}_m} = \frac{1}{2}\int\frac{d^3q}{(2\pi)^3}(1+2n(E_2))\,. \tag{64}$$

Using the relation between $\mu_+$ and $\bar{\mu}_+$ given by Eq. (61), we finally get

$$\rho_{\varphi_1} = \frac{\bar{\mu}_+}{g} + \frac{1}{2}\int\frac{d^3q}{(2\pi)^3}\frac{\frac{q^2}{2m}+\bar{\mu}_+}{E_+}\coth(\beta E_+/2), \tag{65}$$

$$\rho_{\varphi_2} = \frac{1}{2}\int\frac{d^3q}{(2\pi)^3}\coth(\beta E_-/2), \tag{66}$$

with

$$E_+ = \sqrt{\left(\frac{q^2}{2m}\right)\left(\frac{q^2}{2m}+2\bar{\mu}_+\right)}, \tag{67}$$

$$E_- = \frac{q^2}{2m} + \bar{\mu}_+ - \bar{\mu}_-. \tag{68}$$

From Eq. (67), it is evident that the Goldstone mode is preserved at finite interaction and temperature, as it should be. Since $\rho_{\varphi_1}$ and $\rho_{\varphi_2}$ are constants, Eqs. (65), (66), (67) and (68) are coupled equations for the variables $\bar{\mu}_+$ and $\bar{\mu}_-$ as functions of the temperature $T$.

Expressions (65) and (66) have the usual ultraviolet divergences of a field theory at $T = 0$. As it is well known, temperature fluctuations are always convergent. The usual way to deal with this divergence is to regularize the integral and then renormalize the bare constants, $\mu_+$, $\mu_-$ and $g$ in order to obtain finite results. A convenient procedure, in the non-relativistic scalar case, is the cut-off technique. If we simply limit the momentum integrals using an ultraviolet cut-off, $0 \le |\vec{q}| \le \Lambda$, the results are obviously $\Lambda$ dependent. However, if we begin the calculations with renormalized constants, $\mu_\pm^R = \mu_\pm + \delta\mu_\pm(\Lambda)$ we can adjust $\delta\mu_\pm(\Lambda)$ to make the result $\Lambda$ independent. At the end, we can safely take the limit $\Lambda \to \infty$. After this procedure, the renormalized expressions read,

$$\rho_{\varphi_1} = \frac{\bar{\mu}_+}{g} + \frac{(m\bar{\mu}_+)^{3/2}}{3\pi^2} + \int\frac{d^3q}{(2\pi)^3}\frac{\frac{q^2}{2m}+\bar{\mu}_+}{E_+(e^{\beta E_+}-1)}, \tag{69}$$

$$\rho_{\varphi_2} = \int\frac{d^3q}{(2\pi)^3}\frac{1}{e^{\beta E_-}-1}. \tag{70}$$

Equation (69) implicitly defines the condensate density $\bar{\varphi}_m(T)$, or equivalently, the effective chemical potential $\bar{\mu}_+(T)$. This equation coincides with that derived from a one-loop effective potential of a single self-interacting field[24]. Moreover, eq. (70) determines the effective Rabi frequency $\Omega_{\rm eff}(T) = (\bar{\mu}_+ - \bar{\mu}_-)/2 = \Delta\bar{\mu}/2$. The critical temperature, $T_c$, is easily computed by fixing $\bar{\mu}_+ = 0$ in Eq. (69), obtaining the usual expression for an ideal gas,

$$T_c = \frac{2\pi}{m\zeta(3/2)^{2/3}}\rho_{\varphi_1}^{2/3}, \tag{71}$$

with $\zeta(3/2) \sim 2.612$. We expect corrections of $T_c$ only at two-loop approximation. Since $E_-$ represents gapped excitations, the integral in eq. (70) can be safely done in the classical limit, obtaining

$$\rho_{\varphi_2} \sim \frac{1}{4\sqrt{2\pi}}(mT)^{3/2}e^{\frac{\Delta\bar{\mu}}{T}}\,. \tag{72}$$

In terms of the constant densities $\rho_{\varphi_2}$, $\rho_{\varphi_1}$ and the critical temperature $T_c$, given by Eq. (71), we obtain from Eq. (72)

$$\Delta\bar{\mu} = 2\Omega_{\rm eff}(T) = T\ln\left[\left(\frac{\rho_{\varphi_1}}{\rho_{\varphi_2}}\right)\left(\frac{T}{T_c}\right)^{3/2}\right]. \tag{73}$$

Since $\Omega_{\rm eff}(T) > 0$, this calculation is valid only for

$$\frac{T}{T_c} > \left(\frac{\rho_{\varphi_2}}{\rho_{\varphi_1}}\right)^{2/3} . \tag{74}$$

Surprisingly, there is a minimum temperature $T_r = (\rho_{\varphi_2}/\rho_{\varphi_1})^{2/3} T_c$ where $\Delta\bar{\mu}(T_r) = 0$, the $O(2)$ symmetry is restored and then, our mean-field analysis in no longer valid for temperatures $T \leq T_r$. Therefore, in order to have the condensate structure given by eqs. (24) and (25) we need to fix $\rho_{\varphi_2}/\rho_{\varphi_1} < 1$ and $T_r < T < T_c$. This interesting reentrance is not an artifact of the classical approximation used to compute the integral in Eq. (70). In fact, by solving the integral exactly using numerical methods, we find the same general behavior. In the next section, we numerically compute the condensate fractions as functions of temperature for different values of the parameters.

## VI. NUMERICAL RESULTS

To compute the condensate density profile we rewrite Eq. (69) in adimensional form. For this, we define the condensate fraction $\rho_c = \bar{\mu}_+/(g\rho_{\varphi_1})$. The adimensional temperature is defined as $\bar{T} = T/T_c$ and we introduce the diluteness parameter $n_{\varphi_1} = \rho_{\varphi_1} a^3$, where $a$ is the s-wave scattering length. Using these definitions, we can write Eq. (69) in the following form,

$$\begin{aligned} 1 = \rho_c &+ \frac{8}{3\pi^{1/2}} n_{\varphi_1}^{1/2} \rho_c^{3/2} \\ &+ \frac{4}{\pi^{1/2}\zeta(3/2)} \bar{T}^{3/2} \int_0^\infty dy y \frac{y^2 + 2\zeta(3/2)^{2/3} n_{\varphi_1}^{1/3} \rho_c \bar{T}^{-1}}{\sqrt{y^2 + 4\zeta(3/2)^{2/3} n_{\varphi_1}^{1/3} \rho_c \bar{T}^{-1}}} \\ &\times \left( e^{y\sqrt{y^2 + 4\zeta(3/2)^{2/3} n_{\varphi_1}^{1/3} \rho_c \bar{T}^{-1}}} - 1 \right)^{-1} . \end{aligned} \tag{75}$$

It is simple to check that the limit $n_{\varphi_1} \to 0$ leads to the ideal gas result $\rho_c = 1 - \bar{T}^{3/2}$. The second term of the r.h.s. of eq. (75) gives the quantum depletion of the condensate, while the third term represents the temperature dependence. Numerically solving Eq. (75) , we can obtain the condensate fraction $\rho_c(\bar{T})$ for different values of the diluteness parameter $n_{\varphi_1}$. From this result, it is simple to compute the condensate fractions for the fields $\phi$ and $\psi$ using Eqs. (24) and (25).

We define the condensate fractions for the fields $\phi$ and $\psi$ as $\rho_\phi^c = |\phi_0|^2/(\rho_{\varphi_1} + \rho_{\varphi_2})$ and $\rho_\psi^c = |\psi_0|^2/(\rho_{\varphi_1} + \rho_{\varphi_2})$, where we chose the total particle density $\rho_\phi + \rho_\psi = \rho_{\varphi_1} + \rho_{\varphi_2}$ to normalize the fractions. Then, we use Eqs. (24) and (25) to relate $\rho_\phi^c$ and $\rho_\psi^c$ with $\rho_c$, given by Eq. (75).

There are two interesting regimes to focus. For $|\nu|/\Omega_R \ll 1$, the condensate fractions become

$$\rho_\phi^c \sim \left(1 - \frac{\rho_{\varphi_2}}{\rho_{\varphi_1}}\right) \left(\frac{|\nu|}{2\Omega_R}\right)^2 \rho_c , \tag{76}$$

$$\rho_\psi^c \sim \left(1 - \frac{\rho_{\varphi_2}}{\rho_{\varphi_1}}\right) \rho_c . \tag{77}$$

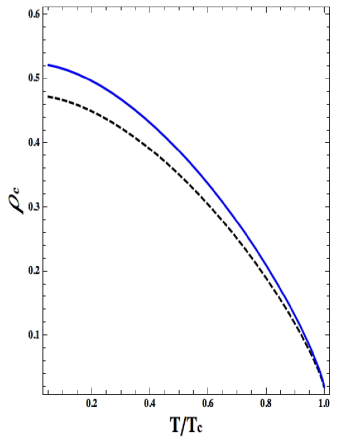


Figura 2: Condensate fractions as functions of the adimensional temperature $\bar{T}$ in the limit $|\nu|/\Omega_R < 1$. The solid line represents $\rho_\psi^c$, given by eq. (77), while the dashed line is $\rho_\phi^c$, given by eq. (76). We have fixed $n_{\varphi_1} = 10^{-5}$, $\rho_{\varphi_2}/\rho_{\varphi_1} = 10^{-1}$ and $|\nu|/2\Omega_R = 0.24$.

The first factor compensates the normalizations of $\rho_{\phi,\psi}^c$ and $\rho_c$. To obtain it, we have considered $\rho_{\varphi_2}/\rho_{\varphi_1} < 1$ and we have dropped terms proportional to $(\rho_{\varphi_2}/\rho_{\varphi_1})^2$. The condensate fraction is determined by the factor $(|\nu|/2\Omega_R)^2$ and the next corrections to eqs. (76) and (77) are terms proportional to $(|\nu|/2\Omega_R)^4$. In Fig. (2) we show the typical profile of both condensates, where we have fixed $n_{\varphi_1} = 10^{-5}$, $\rho_{\varphi_2}/\rho_{\varphi_1} = 10^{-1}$ and $|\nu|/2\Omega_R = 0.24$. Note that $\rho_\phi^c$ is strongly suppressed by the factor $\nu/\Omega_R$ and tends to disappear in the limit $|\nu| \to 0$. An interesting observation is that the factor $|\nu|/\Omega_R$ is not corrected by temperature fluctuations. This is a direct consequence of the $O(2)$ symmetry of the two-body interaction.

In the opposite regime $\Omega_R/|\nu| \ll 1$, the condensate densities of both species are essentially equal, with small corrections, given by

$$\rho_{\phi_0}^c \sim \frac{1}{2}\left(1 - \frac{\rho_{\varphi_2}}{\rho_{\varphi_1}}\right)\left(1 - \frac{\Omega_R}{|\nu|}\right)\rho_c , \tag{78}$$

$$\rho_{\psi_0}^c \sim \frac{1}{2}\left(1 - \frac{\rho_{\varphi_2}}{\rho_{\varphi_1}}\right)\left(1 + \frac{\Omega_R}{|\nu|}\right)\rho_c , \tag{79}$$

where we have discarded corrections of order $(\Omega_R/|\nu|)^2$. We show these curves in Fig. (3) for $n_{\varphi_1} = 10^{-5}$, $\rho_{\varphi_2}/\rho_{\varphi_1} = 10^{-1}$ and $\Omega_R/|\nu| = 0.02$.

## VII. DISCUSSION

We have addressed the problem of equilibrium properties of a uniform mixture of two bosonic fields in the presence of Josephson-type interactions. We have considered a quantum field theory built by two non-relativistic complex bosonic fields with general two-body local interactions. We have focused on a particular symmetry

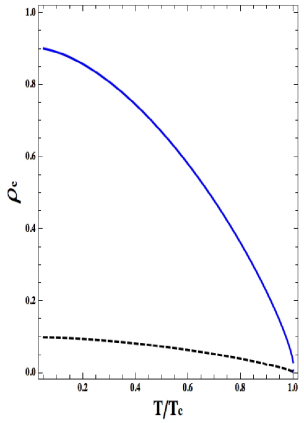


Figura 3: Condensate fractions as functions of the adimensional temperature $\bar{T}$ in the limit $\Omega_R/|\nu| < 1$. The solid line represents $\rho^c_\psi$, given by eq. (79), while the dashed line is $\rho^c_\phi$, given by eq. (78). We have fixed $n_{\varphi_1} = 10^{-5}$, $\rho_{\varphi_2}/\rho_{\varphi_1} = 10^{-1}$ and $\Omega_R/|\nu| = 0.02$.

point, in which, in addition to the $U(1) \otimes U(1)$ phase symmetry, threre is an emergent $O(2)$ symmetry, related with rotations in the iso-spin space $(\phi, \psi)$. In the absence of Josephson interactions, this model generally presents phase separation. However, interactions that explicitly breaks $U(1) \otimes U(1) \otimes O(2)$ symmetry changes this scenario.

We have minimally perturbed this model by considering the effect of Josephson couplings that unbalance the species population by transferring charge from one species to the other. These interactions are parametrized by the Rabi frequency $\Omega_R$ and the detuning $\nu$. By making a rotation in the iso-spin space, $(\phi, \psi) \to (\varphi_1, \varphi_2)$, we have shown that there is a special direction where the $U(1) \otimes U(1)$ phase symmetry is recovered and only one of the bosonic species (say $\varphi_1$) could eventually condensate in this framework. In this basis, the particle number of each bosonic species $\rho_{\varphi_1}$ and $\rho_{\varphi_2}$ is conserved independently. Of course, the $O(2)$ symmetry is still broken, provided the difference between chemical potentials $\Delta\mu = \mu_+ - \mu_- = 2\Omega_{\text{eff}} \neq 0$.

In the $(\varphi_1, \varphi_2)$ basis, it is simpler to compute fluctuations. Specifically, we have computed finite temperature one-loop effective action (the Gibbs free energy) as a function of the order parameter and temperature. In this way, by minimizing the free energy we have obtained the condensate fraction. Interestingly, we have found a reentrance of the $O(2)$ symmetry at a specific temperature $T_r$, for which $\Omega_{\text{eff}}(T_r) = 0$. This temperature essentially depends on the particle number imbalance between the two species, $\rho_{\varphi_1}/\rho_{\varphi_2}$. Below this temperature, the mean-field approach is unstable under thermal fluctuations. Thus, our approach is valid for $T_r < T < T_c$.

To obtain the condensate profiles of the original fields, we turned back the rotation to the original bases $(\phi, \psi)$. This rotation only depends on the ratio $\Omega_R/|\nu|$. It is interesting to note that, due to the $O(2)$ symmetry of the two-body interaction, fluctuations only renormalize the effective Rabi frequency $\Omega_{\text{eff}} = \sqrt{\Omega_R^2 + |\nu|^2}$, while the ratio $\Omega_R/|\nu|$ remains unaffected. Thus, the iso-spin rotation coefficients are temperature independent.

In Figures (2) and (3) we show the condensate profiles of the $\psi$ and $\phi$ species as functions of temperature for different values of the parameter $\Omega_R/|\nu|$. We have shown that, for a temperature interval $T_r < T < T_c$, both bosonic species condensate and the relatives phases are locked by the detuning phase $\alpha_\nu$. We also showed that the ratio between the condensates essentially depends on the temperature-independent parameter $\Omega_R/|\nu|$. We clearly see that, for $|\nu|/\Omega_R \to 0$, only one condensate survives, while in the opposite limit $\Omega_R/|\nu| \to 0$, both condensates are essentially equal, with small corrections of order $\Omega_R/|\nu|$.

The results presented in this paper are valid provided the two-body interaction has iso-spin rotation invariance. Consider, for instance, a small deviation from de $O(2)$ model, $g_\phi = g_\psi = g$, but $g_{\psi\phi} = g + \Delta g$. Upon rotation to the $(\varphi_1, \varphi_2)$ bases, a term proportional to $\Delta g(\varphi_1^*\varphi_1^*\varphi_2\varphi_2)$ will be generated. Thus, even though we have ignored this type of terms in the original model, they will be generated in a more general two-body interaction case. Thus, for $\Delta g \neq 0$, there is no iso-spin direction in which the $U(1) \otimes U(1)$ symmetry is recovered. This fact makes the study of quantum and thermal fluctuations more involved. We hope to report on this issue shortly.

[1] P. Soltan-Panahi, Multi-Component Quantum Gases in Optical Lattices, Verlag Dr. Hut, (2010).
[2] P. Soltan-Panahi, J. Struck, P. Hauke, A. Bick, W. Plenkers, G. Meineke, C. Becker, P. Windpassinger, M. Lewenstein and K. Sengstock, Nature Physics **7**, 434 (2011).
[3] M. R. Andrews, C. G. Townsend, H. J. Miesner, D. S. Durfee, D. M. Kurn, W. Ketterle, Science **175**, 637 (1997).
[4] Bryce Gadway, Daniel Pertot, René Reimann, and Dominik Schneble, Phys. Rev. Lett. **105**, 045303 (2010).
[5] M. Iskin, C. A. R. Sa de Melo, Phys. Rev. Lett. **97**, 100404 (2006); Phys. Rev. **A 83**, 045602 (2011).
[6] C. J. Myatt, E. A. Burt, R. W. Ghrist, E. A. Cornell and C. E. Wieman, Phys. Rev. Lett. **78**, 586 (1997).
[7] Hall, D. S., M. R. Matthews, J. R. Ensher, C. E. Wieman, and E. A. Cornell, Phys. Rev. Lett. **81**, 1539 (1998).
[8] Hall, D. S., M. R. Matthews, C. E. Wieman, and E. A. Cornell, Phys. Rev. Lett. **81**, 1543 (1998).
[9] Y. J. Lin, K. Jimenez-Garcia, and I.B. Spielman, Nature

**471**, 83 (2011).
[10] T. Zibold, E. Nicklas, C. Gross, and M. K. Oberthaler, Phys. Rev. Lett. **105**, 204101 (2010).
[11] A. J. Leggett and F. Sols, Found. Phys. **21**, 353 (1991).
[12] I. Zapata, F. Sols, and A. J. Leggett, Phys. Rev.**A 57**, R28 (1998).
[13] Josephson, B. D., Phys. Lett. **1**, 251 (1962).
[14] A. Barone, Quantum Mesoscopic Phemena and Mesoscopic Devices in Microelectronics, NATO Sience Series C: Mathematical and Physical Sciences **559**, edited by I. O. Kulik (Kluwer Academic, Dordrecht/Boston), 301 (2000).
[15] Anthony J. Leggett, Reviews of Modern Physics, **73**, 307 (2001).
[16] M. Albiez, R. Gati, J. Folling, S. Hunsmann, M. Cristiani, M. K. Oberthaler, Phys. Rev. Lett. **95**, 010402 (2005).
[17] S. B. Papp, J. M. Pino and C. E. Wieman, Phys. Rev. Lett. **101**, 040402 (2008).
[18] R. Gati and M. K. Oberthaler, J. Phys. B: At. Mol. Opt. Phys. **40**, 10, R61 (2007).
[19] S. Levy, E. Lahoud, I. Shomroni, J. Steinhauer, Nature **449**, 579 (2007).
[20] G. Mazzarella, B. Malomed, L. Salasnich, M. Salerno and F. Toigo, J. Phys. B: At. Mol. Opt. Phys. **44**, 035301(2011).
[21] A. Smerzi, S. Fantoni, S. Giovanazzi, and S. R. Shenoy, Phys. Rev. Lett. **79**, 4950 (1997).
[22] S. Raghavan, A. Smerzi, S.Fantoni, and S. R. Shenoy, Phys. Rev. **A 59**, 620 (1999).
[23] M. Salerno, Laser Physics **4**, 620 (2005).
[24] M. B. Pinto, R. O. Ramos and F. F. de Souza Cruz, Phys. Rev. **A74**, 033618 (2006).
[25] M. Moshe, J. Zinn-Justin, Physics Reports **385**, 69 (2003).
[26] Chih-Chun Chien, F. Cooper and E. Timmermans, Phys. Rev. **A86**, 023634 (2012).
[27] M.A. Garcia-March, G. Mazzarella, L. DellAnna, B. Juliá-Díaz, L. Salasnich, A. Polls, Phys. Rev. **A 89**, 063607 (2014).
[28] M. R. Matthews et al, Phys. Rev. Lett. **81**, 243 (1998).